

# CS 170 Notes: Fast Fourier Transform

## 1 Discrete Fourier Transform

### 1.1 Definition

The Discrete Fourier Transform (DFT) is a linear operation on vectors that is widely used in signal processing, image processing, speech recognition, data compression, and many other fields. As a linear operator, the DFT can be represented as a matrix  $M_n(\omega)$  when operating on a vector of length  $n$ .  $M_n(\omega)$  is defined as the  $n \times n$  matrix

$$\begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \omega & \omega^2 & \dots & \omega^{n-1} \\ 1 & \omega^2 & \omega^4 & \dots & \omega^{2(n-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{n-1} & \omega^{2(n-1)} & \dots & \omega^{(n-1)(n-1)} \end{pmatrix}$$

where  $\omega$  is the *primitive*  $n$ th root of unity, equal to the complex number  $e^{(2\pi i/n)}$ . Note that there are  $n$  different  $n$ th roots of unity:  $1, \omega, \omega^2, \dots, \omega^{n-1}$ , where  $\omega$  is the primitive root.

Some important points:

- (1) All the matrix entries are  $n$ th roots of unity (since an integer power of a root of unity is also a root of unity).
- (2) The inverse matrix of  $M_n(\omega)$  is  $\frac{1}{n}M_n(\omega^{-1})$ .
- (3)  $n$  does **not** need to be a power of 2. The DFT is valid for any positive integer  $n$ . Only a particular implementation of the FFT (described below) requires  $n$  to be a power of 2.
- (4) Straightforward matrix-vector multiplication requires  $O(n^2)$  operations. We will see that the FFT below is a way of computing the DFT faster, in  $O(n \log n)$  operations.

(4) If the input vector is  $v = \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_{n-1} \end{pmatrix}$ , then the output vector  $M_n(\omega) \times v$  is  $\begin{pmatrix} A(\omega^0) \\ A(\omega^1) \\ \vdots \\ A(\omega^{n-1}) \end{pmatrix}$ ,

where  $A(x) = a_0 + a_1x + \dots + a_{n-1}x^{n-1}$ . Thus, the DFT takes a vector in the coefficient representation of a polynomial and transforms it to an evaluation representation of the polynomial (evaluated at the  $n$ th roots of unity).

## 1.2 Examples

### 1.2.1 DFT of a Vector

When  $n = 4$ ,  $\omega = e^{(2\pi i/4)} = i$ , so the DFT matrix  $M_4(\omega)$  is

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{pmatrix}.$$

Therefore, the DFT of  $\begin{pmatrix} 1 \\ 0 \\ -1 \\ 2 \end{pmatrix}$  is

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{pmatrix} \times \begin{pmatrix} 1 \\ 0 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 - 2i \\ -2 \\ 2 + 2i \end{pmatrix}.$$

### 1.2.2 Polynomial Evaluation

Similarly, to evaluate  $A(x) = 3 - 4x + x^2 + 2x^3$  at  $x = \omega^0, \omega^1, \omega^2$ , and  $\omega^3$ , we take the DFT of

$$\begin{pmatrix} 3 \\ -4 \\ 1 \\ 2 \end{pmatrix} \text{ and obtain } \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{pmatrix} \times \begin{pmatrix} 3 \\ -4 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 - 6i \\ 6 \\ 2 + 6i \end{pmatrix}.$$

Thus,  $A(\omega^0) = 2$ ,  $A(\omega^1) = 2 - 6i$ ,  $A(\omega^2) = 6$ , and  $A(\omega^3) = 2 + 6i$ , or equivalently,  $A(1) = 2$ ,  $A(i) = 2 - 6i$ ,  $A(-1) = 6$ , and  $A(-i) = 2 + 6i$ .

### 1.2.3 Polynomial Multiplication

To multiply the polynomials  $A(x) = 2 + x - 4x^2 + x^3$  and  $B(x) = 3 - x^2$ , we first note that their product  $C(x) = A(x) \times B(x)$  is an order 5 polynomial ( $x^5$  will be the highest order term in their product), so we must choose  $n \geq 6$ . This is because if the order of a polynomial is  $d$ , it must be evaluated at no less than  $d + 1$  points to completely determine its coefficients. Therefore, we choose  $n = 6$  so we can evaluate  $A(x)$  and  $B(x)$  at  $x = \omega^0, \omega^1, \dots, \omega^5$ .

We first determine the DFT matrix, noting  $\omega = e^{2\pi i/6} = \frac{1}{2}(1 + i\sqrt{3})$ :

$$M_6(\omega) = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \frac{1}{2}(1 + i\sqrt{3}) & \frac{1}{2}(-1 + i\sqrt{3}) & -1 & \frac{1}{2}(-1 - i\sqrt{3}) & \frac{1}{2}(1 - i\sqrt{3}) \\ 1 & \frac{1}{2}(-1 + i\sqrt{3}) & \frac{1}{2}(-1 - i\sqrt{3}) & 1 & \frac{1}{2}(-1 + i\sqrt{3}) & \frac{1}{2}(-1 - i\sqrt{3}) \\ 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & \frac{1}{2}(-1 - i\sqrt{3}) & \frac{1}{2}(-1 + i\sqrt{3}) & 1 & \frac{1}{2}(-1 - i\sqrt{3}) & \frac{1}{2}(-1 + i\sqrt{3}) \\ 1 & \frac{1}{2}(1 - i\sqrt{3}) & \frac{1}{2}(-1 - i\sqrt{3}) & -1 & \frac{1}{2}(-1 + i\sqrt{3}) & \frac{1}{2}(1 + i\sqrt{3}) \end{pmatrix}.$$

Then, the DFT of the coefficients of  $A(x)$  is

$$M_6(\omega) \times \begin{pmatrix} 2 \\ 1 \\ -4 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{1}{2}(7 - 3i\sqrt{3}) \\ \frac{1}{2}(9 + 5i\sqrt{3}) \\ -4 \\ \frac{1}{2}(9 - 5i\sqrt{3}) \\ \frac{1}{2}(7 + 3i\sqrt{3}) \end{pmatrix}.$$

Similarly, the DFT of the coefficients of  $B(x)$  is

$$M_6(\omega) \times \begin{pmatrix} 3 \\ 0 \\ -1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ \frac{1}{2}(7 - i\sqrt{3}) \\ \frac{1}{2}(7 + i\sqrt{3}) \\ 2 \\ \frac{1}{2}(7 - i\sqrt{3}) \\ \frac{1}{2}(7 + i\sqrt{3}) \end{pmatrix}.$$

Now, we want to evaluate the product  $C(x)$  at the 6th roots of unity, so we multiply (component-wise) the two results above to obtain

$$\begin{pmatrix} 0 \\ \frac{1}{4}(7 - 3i\sqrt{3})(7 - i\sqrt{3}) \\ \frac{1}{4}(9 + 5i\sqrt{3})(7 + i\sqrt{3}) \\ -8 \\ \frac{1}{4}(9 - 5i\sqrt{3})(7 - i\sqrt{3}) \\ \frac{1}{4}(7 + 3i\sqrt{3})(7 + i\sqrt{3}) \end{pmatrix} = \begin{pmatrix} 0 \\ 10 - 7i\sqrt{3} \\ 12 + 11i\sqrt{3} \\ -8 \\ 12 - 11i\sqrt{3} \\ 10 + 7i\sqrt{3} \end{pmatrix}.$$

This result is the DFT of the coefficient vector of  $C(x)$ . Therefore, to obtain the coefficient vector, we must take the inverse DFT on our result. We have

$$\frac{1}{6}M_6(\omega^{-1}) = \frac{1}{6} \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \frac{1}{2}(-1 + i\sqrt{3}) & \frac{1}{2}(-1 - i\sqrt{3}) & -1 & \frac{1}{2}(1 - i\sqrt{3}) & \frac{1}{2}(1 + i\sqrt{3}) \\ 1 & \frac{1}{2}(-1 - i\sqrt{3}) & \frac{1}{2}(1 - i\sqrt{3}) & 1 & \frac{1}{2}(-1 - i\sqrt{3}) & \frac{1}{2}(1 - i\sqrt{3}) \\ 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & \frac{1}{2}(1 - i\sqrt{3}) & \frac{1}{2}(-1 - i\sqrt{3}) & 1 & \frac{1}{2}(1 - i\sqrt{3}) & \frac{1}{2}(-1 - i\sqrt{3}) \\ 1 & \frac{1}{2}(1 + i\sqrt{3}) & \frac{1}{2}(1 - i\sqrt{3}) & -1 & \frac{1}{2}(-1 - i\sqrt{3}) & \frac{1}{2}(-1 + i\sqrt{3}) \end{pmatrix}.$$

This gives an inverse DFT of:

$$\frac{1}{6}M_6(\omega^{-1}) \times \begin{pmatrix} 0 \\ 10 - 7i\sqrt{3} \\ 12 + 11i\sqrt{3} \\ -8 \\ 12 - 11i\sqrt{3} \\ 10 + 7i\sqrt{3} \end{pmatrix} = \begin{pmatrix} 6 \\ 3 \\ -14 \\ 2 \\ 4 \\ -1 \end{pmatrix}.$$

Thus, we have our product  $C(x) = 6 + 3x - 14x^2 + 2x^3 + 4x^4 - x^5$ .

## 2 Fast Fourier Transform

### 2.1 Definition

The Fast Fourier Transform (FFT) is a divide-and-conquer algorithm for performing the DFT in  $O(n \log n)$  time instead of the  $O(n^2)$  time required by straightforward matrix-vector multiplication. It is one of the most widely used computer algorithms today because of its speed and vast number of applications (speech recognition, image processing, data compression, polynomial multiplication, etc).

For many practical (and unfortunately complex) implementations of the FFT, the input vector size  $n$  can be any positive integer. However, the particular FFT algorithm given in lecture and the textbook restricts  $n$  to a power of 2 since it repeatedly divides the problem into two subproblems. See the top of page 64 in the textbook for the polynomial formulation of the FFT. It evaluates a polynomial  $A(x)$  at the  $n$ th roots of unity (this is the same as the DFT because the FFT is just a fast way of doing the DFT) by splitting the polynomial into its even  $A_e(x)$  and odd  $A_o(x)$  terms, such that  $A(x) = A_e(x^2) + xA_o(x^2)$ . It then recursively evaluates the two smaller polynomials  $A_e(x)$  and  $A_o(x)$  at the  $(n/2)$ th roots of unity (since the squares of the  $n$ th roots of unity become the  $(n/2)$ th roots of unity). It recurses until it reaches the base case of evaluating  $n$  polynomials at the single point  $x = \omega^0 = 1$ . Even though this algorithm seems entirely different from the DFT matrix multiplication, page 67 of the textbook shows how the FFT is simply a divide-and-conquer approach to multiplying by  $M_n(\omega)$ .

For more information on a real-world implementation of the FFT, see the “Fastest Fourier Transform in the West” at [www.fftw.org](http://www.fftw.org).

### 2.2 Example

Let's determine the DFT of  $\begin{pmatrix} 3 \\ -4 \\ 1 \\ 2 \end{pmatrix}$ . Although we could use the straightforward matrix-vector multiplication method, let's use the FFT.

The input vector is the coefficient vector of  $A(x) = 3 - 4x + x^2 + 2x^3$ , which we want to evaluate at  $x = \omega^0, \omega^1, \omega^2, \text{ and } \omega^3$ . The even terms of  $A(x)$  are 3 and  $x^2$ , so  $A_e(x) = F(x) = 3 + x$ . Similarly, the odd terms of  $A(x)$  are  $-4x$  and  $2x^3$ , so  $A_o(x) = G(x) = -4 + 2x$ . You can check that  $A(x) = A_e(x^2) + xA_o(x^2) = F(x^2) + xG(x^2)$  as required.

For the next level of recursion, we must evaluate  $F(x)$  and  $G(x)$  at  $x = \omega^0$  and  $\omega^2$ . To do this recursively, we find the even and odd parts of  $F(x)$  and  $G(x)$ :  $F_e(x) = 3$ ,  $F_o(x) = 1$ ,  $G_e(x) = -4$ , and  $G_o(x) = 2$ . Now we must evaluate four polynomials ( $F_e$ ,  $F_o$ ,  $G_e$ , and  $G_o$ ) at  $x = \omega^0 = 1$ . However, these polynomials are constant, so we have reached our base case.

With the recursion finished, we can now evaluate  $F(x)$  and  $G(x)$  at  $x = \omega^0$  and  $\omega^2$ . Note that  $\omega = e^{2\pi i/4} = i$  when  $n = 4$  and that  $\omega^4 = \omega^0$ .

$$\begin{aligned} F(\omega^0) &= F_e(\omega^0) + \omega^0 F_o(\omega^0) = 3 + 1 \cdot 1 = 4 \\ F(\omega^2) &= F_e(\omega^4) + \omega^2 F_o(\omega^4) = 3 - 1 \cdot 1 = 2 \\ G(\omega^0) &= G_e(\omega^0) + \omega^0 G_o(\omega^0) = -4 + 1 \cdot 2 = -2 \end{aligned}$$

$$G(\omega^2) = G_e(\omega^4) + \omega^2 G_o(\omega^4) = -4 - 1 \cdot 2 = -6$$

Finally, we use these results to evaluate  $A(x)$  at  $x = \omega^0, \omega^1, \omega^2$ , and  $\omega^3$ . Note that  $\omega^4 = \omega^0$  and  $\omega^6 = \omega^2$ .

$$A(\omega^0) = F(\omega^0) + \omega^0 G(\omega^0) = 4 + 1 \cdot (-2) = 2$$

$$A(\omega^1) = F(\omega^2) + \omega^1 G(\omega^2) = 2 + i \cdot (-6) = 2 - 6i$$

$$A(\omega^2) = F(\omega^4) + \omega^2 G(\omega^4) = 4 + (-1) \cdot (-2) = 6$$

$$A(\omega^3) = F(\omega^6) + \omega^3 G(\omega^6) = 2 - i \cdot (-6) = 2 + 6i$$

Thus, the DFT is  $\begin{pmatrix} 2 \\ 2 - 6i \\ 6 \\ 2 + 6i \end{pmatrix}$ .