1. (a) Show that the language of all strings (over \( \Sigma = \{0, 1\} \)) which do not contain the substring 011 is a regular language by exhibiting a DFA/NFA that recognizes it.

(b) Suppose \( A \) is a regular language over \( \Sigma \). Let \( \Sigma' \) be some alphabet such that \( \Sigma \subset \Sigma' \). Using only DFAs, show that \( A \) is a regular language over \( \Sigma' \).

2. (Sipser 1.24) For any string \( w = w_1w_2 \ldots w_n \), the reverse of \( w \), written \( w^R \), is the string \( w \) in reverse order, \( w_n \ldots w_2w_1 \). For any language \( A \), let \( A^R = \{ w^R \mid w \in A \} \). Show that if \( A \) is regular, so is \( A^R \).

3. (Sipser 1.25) Let

\[ \Sigma_3 = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}, \ldots, \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \right\} \]

\( \Sigma_3 \) contains all size 3 columns of 0s and 1s. A string of symbols in \( \Sigma_3 \) gives three rows of 0s and 1s. Consider each row to be a binary number and let

\[ B = \{ w \in \Sigma_3^* \mid \text{the bottom row of } w \text{ is the sum of the top two rows} \} \]

For example,

\[ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \in B, \quad \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \notin B. \]

Show that \( B \) is regular. (Hint: Working with \( B^R \) is easier. You may assume the result claimed in the last problem.)

4. (Sipser 1.28) Let

\[ \Sigma_2 = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}. \]
Here, $\Sigma_2$ contains all columns of 0s and 1s of height two. Consider the top and bottom rows to be strings of 0s and 1s and let

$$E = \{ w \in \Sigma_2^* | \text{the bottom row of } w \text{ is the reverse of the top row of } w \}.$$ 

Show that $E$ is not regular.

5. Let $x$ and $y$ be strings and let $L$ be any language. We say that $x$ and $y$ are distinguishable by $L$ if some string $z$ exists whereby exactly one of the strings $xz$ and $yz$ is a member of $L$. Otherwise, if for every string $z$, $xz \in L$ if and only if $yz \in L$, we say that $x$ and $y$ are indistinguishable by $L$. If $x$ and $y$ are indistinguishable by $L$ we write $x \equiv_L y$.

(a) (Sipser 1.34) Show that $\equiv_L$ is an equivalence relation.
(b) Suppose we have a set of strings $\{x_1, x_2, \ldots, x_k\}$ such that $x_i$ and $x_j$ are distinguishable by $L$ for $i \neq j$. Show that any DFA which recognizes $L$ must have at least $k$ states.