Read these instructions carefully:

1. This is a closed-book exam. You have 80 minutes to complete the exam.

2. Calculators are permitted.

3. You are allowed a two-sided, letter-size cheat sheet.

4. Answer the multiple choice questions by circling the correct answer(s). No partial credit will be given for multiple choice questions. Incorrect answers will receive a negative score (−2), so if you do not know the answer, you should not guess.

5. Write your answers to the other questions in the spaces provided below them. If you don’t have enough space, continue on the back of the page and state clearly that you have done so. Show all your work to receive partial credit.
1. (20 pt) **Multiple choice questions.** Circle your choice. Incorrect answers will receive a negative score (−2), so if you do not know the answer, you should not guess.

(a) (4 pt) Let $A$ and $B$ be events with $\Pr(A) = \frac{3}{4}$ and $\Pr(B) = \frac{1}{3}$. Which of the following is true? (Circle one choice.)

i. $\frac{1}{12} \leq \Pr(A \cap B) \leq \frac{1}{3}$.

ii. $\frac{1}{3} \leq \Pr(A \cap B) \leq \frac{3}{4}$.

iii. $\frac{1}{4} \leq \Pr(A \cap B) \leq \frac{3}{4}$.

iv. $\frac{1}{4} \leq \Pr(A \cap B) \leq \frac{1}{3}$.

v. None of the above.

(b) (4 pt) Suppose that at least one of the events $X_1, X_2, \ldots, X_n$ is certain to occur, but certainly no more than two occur. If $\Pr(X_i) = p$ for all $i = 1, \ldots, n$ and $\Pr(X_i \cap X_j) = q$ for all $i \neq j$, which of the following is true? (Circle one choice.)

i. $p \geq \frac{1}{n}$ and $q \leq \frac{2}{n}$.

ii. $p \geq \frac{1}{n}$ and $q \geq \frac{2}{n}$.

iii. $p \geq \frac{2}{n}$ and $q \leq \frac{1}{n}$.

iv. $p \geq \frac{2}{n}$ and $q \geq \frac{1}{n}$.

v. None of the above.

(c) (4 pt) Let $A$ and $B$ be independent events. Circle all true statements.

i. $\Pr(A \cap B \mid C) = \Pr(A \mid C)\Pr(B \mid C)$.

ii. $\Pr(A \cap B \cap C) = \Pr(A \cap C)\Pr(B \cap C)$.

iii. $\overline{A}$ and $B$ are independent.

iv. $\overline{A}$ and $\overline{B}$ are independent.

v. $\Pr(A \cup B) = \Pr(A) + \Pr(B)$. 


(d) (4 pt) Let $X$ be a random variable that takes values in the interval $(-10, 10)$ and satisfies $P(X = x) = P(X = -x)$, for all $x \in (-10, 10)$. Circle all true statements.

i. $E[X^3] \geq (E[X])^3$.

ii. $E[X^4] \geq (E[X])^4$.

iii. $E[X - X^2] \leq E[X] - (E[X])^2$.

iv. $E[X] = 0$.

v. $\text{Var}[X] \geq 0$.

(e) (4 pt) Consider a random bit generator that produces a 1 with probability $p$ and a 0 with probability $(1 - p)$. Suppose you run this random bit generator 100 times, and let $X_i$ denote the $i$th outcome. Define $X = \sum_{i=1}^{100} X_i$. Given that $X = 2$, what is the probability that $X_1 = 1$? (Circle one choice.)

i. $p$.

ii. $\frac{p}{\binom{100}{2} p^2 (1-p)^{98}}$.

iii. $\frac{1}{50}$.

iv. $\frac{1}{100}$.

v. None of the above.
2. (20 pt) **Short computational problems.**

(a) (10 pt) Let $X_1$ and $X_2$ be independent random variables with $\mathbb{P}(X_i = k) = (1 - p_i)p_i^k$, for $k = 0, 1, 2, \ldots$. Compute the probability $\mathbb{P}(X_1 < X_2)$. 
(b) (5 pt) A monkey types on a keyboard that has only 26 lowercase letters \{a,b,c,\ldots,x,y,z\} and nothing else. At each keystroke, each of the 26 letters is equally likely to be hit. The monkey types 1000 letters. What is the expected number of times the word “evolution” appears in this text?

(c) (5 pt) In class, we discussed Karger’s randomized algorithm for finding a min-cut in an undirected graph \(G\) with \(n\) vertices. As a direct consequence of its analysis, one can show that the total number of min-cuts that \(G\) can have is upper bounded by \(\binom{n}{2}\). Prove this result and give an example \(G\) with exactly \(\binom{n}{2}\) min-cuts. (This proves that \(\binom{n}{2}\) is a tight upper bound.)
3. (30 pt) **Tail bounds.**

(a) (6 pt) State and prove Markov’s inequality.

(b) (5 pt) State and prove Chebyshev’s inequality.
(c) (5 pt) In the remainder of this problem, consider a standard six-sided die and let $X$ denote the number of times that 6 shows up over $n$ throws of the die. Compute an upper bound on $P(X \geq \frac{n}{2})$ using Markov’s inequality.

(d) (6 pt) Compute an upper bound on $P(X \geq \frac{n}{2})$ using Chebyshev’s inequality.
(e) (8 pt) Compute the best upper bound on \( \mathbb{P}(X \geq \frac{n}{2}) \) using a Chernoff bound. (HINT: You need to find \( t \) that gives the best Chernoff bound.)
4. (30 pt) **Number of cycles in a random permutation.** A permutation \( \pi : \{1, \ldots, n\} \to \{1, \ldots, n\} \) can be represented as a set of cycles as follows. Assign one vertex to each number \( i = 1, \ldots, n \). If the permutation maps \( i \) to \( \pi(i) \), then a directed arc is drawn from vertex \( i \) to vertex \( \pi(i) \). This leads to a graph that is a set of disjoint cycles. Note that if the permutation maps \( i \) to \( i \), then the cycle is a self-loop from \( i \) to \( i \). In what follows, consider a random permutation \( \pi \) of \( n \) numbers. Let \( X \) denote the total number of cycles in \( \pi \).

(a) (5 pt) Find \( P(X = 1) \), the probability that a random permutation consists of a single cycle.

(b) (8 pt) Find \( E[X] \) for \( n = 2 \) and \( n = 3 \).
(c) (10 pt) Define $Y_i := 1/(\text{length of the cycle containing } i)$. Find $\mathbb{P}[Y_i = \frac{1}{k}]$ and $\mathbb{E}[Y_i]$. 
(d) (4 pt) Find an equation that relates $X$ and $Y_i, i = 1, ..., n$.

(e) (3 pt) Compute $E[X]$. 