1. **(Balls and Bins.)**
Consider the event that every bin receives exactly $k$ balls when $kn$ balls are thrown randomly into $n$ bins.

(a) Determine the exact probability of this event.

(b) Compute the probability under the Poisson approximation.

(c) The value upon dividing the expression in (b) by that in (a) equals the probability that a Poisson random variable with parameter $\lambda$ takes on some value $r$. Explain briefly but precisely why this quotient matches a Poisson distribution. Also, what are the values of $\lambda$ and $r$?
2. **(A Threshold for 4-Cycles.)**

In this problem we will see that the value \( p = 1/n \) is a “threshold” for the property that a random graph in the \( G_{n,p} \) model contains a cycle of length 4. That is, we will prove that

\[
P[G \text{ contains a cycle of length 4}] \rightarrow \begin{cases} 
0 & \text{if } p = o\left(\frac{1}{n}\right), \\
1 & \text{if } p = \omega\left(\frac{1}{n}\right) 
\end{cases} \quad \text{as } n \rightarrow \infty.
\]

(a) Let the random variable \( X \) denote the number of cycles of length 4 in \( G \). Write down the expectation of \( X \) as a function of \( n \) and \( p \).

(b) Show that \( E[X] \rightarrow 0 \) for \( p = o\left(\frac{1}{n}\right) \), and that \( E[X] \rightarrow \infty \) for \( p = \omega\left(\frac{1}{n}\right) \).

(c) Deduce from part (b) that \( P[G \text{ contains a cycle of length 4}] \rightarrow 0 \) for \( p = o\left(\frac{1}{n}\right) \).

(d) Show that \( \text{Var}[X] = O\left(n^4p^4 + n^6p^7 + n^5p^6\right) \).

(e) Deduce from parts (b) and (d) that \( P[G \text{ contains a cycle of length 4}] \rightarrow 1 \) for \( p = \omega\left(\frac{1}{n}\right) \).
3. (Finding a Good 3-Coloring.)
A 3-coloring of an undirected graph $G = (V, E)$ is an assignment of colors red, green, or blue to every vertex in the graph. An edge is well-colored if the two endpoints of the edge are assigned distinct colors. In the MAX3COLOR problem, we are given an undirected graph $G$ and asked to find a 3-coloring with the maximum possible number of well-colored edges.

(a) Using the probabilistic method, show that every graph has a 3-coloring with at least $\frac{2}{3}|E|$ well-colored edges.

(b) Using the method of conditional probabilities, we may derive a deterministic polynomial-time algorithm that, given $G$, outputs a 3-coloring with at least $\frac{2}{3}|E|$ well colored edges. Explain how to compute

$$\mathbb{E}[X|c_1, c_2, ..., c_i]$$

where $c_1, ..., c_i$ are the assignments of colors to the first $i$ vertices.

(c) Explain briefly why the above algorithm is guaranteed to output a 3-coloring with at least $\frac{2}{3}|E|$ well-colored edges.
4. The lollipop graph on \( n \) vertices is a clique on \( n/2 \) vertices connected to a path on \( n/2 \) vertices, as shown in Figure 1. The node \( u \) connects the clique to the path, and node \( v \) is at the opposite end of the path from node \( u \).

![Lollipop Graph Diagram](image)

Figure 1: On four vertices, there are three possible cycles of length four.

(a) Show that the expected covering time of a random walk starting at \( v \) is \( O(n^2) \). (Hint: Break up the walk into walks on different segments of the graph, and use the fact that from the vertex just to the right of \( u \), it takes \( \Theta(n) \) expected time to reach either \( u \) or \( v \).)

(b) Show that the expected covering time of a random walk starting at \( u \) is \( O(n^3) \).