**Definition**

The generating function for a finite sequence of elements $a_0, a_1, \ldots, a_n$ is the polynomial

$$f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$$

and for infinite sequences the generating function is similarly defined with a sum to infinity. Generating functions allow all the usual operations on power series, like addition, multiplication and formal division (like long division, starting from the lowest power). They also support formal differentiation and integration.

**Series Sums**

Generating functions make it extremely easy to sum a series by evaluating the function at 1, or to evaluate the sum of the series times powers of a constant.

$$f(1) = \sum_i a_i \quad f(b) = \sum_i a_ib^i$$

**Binomial Expansion**

The expansion of $(1 + x)^n$ is

$$f(x) = (1 + x)^n = \sum_{i=0}^{n} \binom{n}{k} x^k \quad \text{so} \quad f(1) = 2^n = \sum_{i=0}^{n} \binom{n}{k}$$

**Geometric Series**

The expansion of $1/(1 - x)^n$ is the infinite power series $1 + x + x^2 + x^3 + \cdots$. Evaluating at $b$ gives the usual formula for the sum of a geometric series.

$$\frac{1}{1 - b} = \sum_{i=0}^{\infty} b^i$$

To get the sum of a finite number of terms we subtract a shifted geometric series

$$\sum_{i=0}^{n} b^i - \sum_{i=0}^{\infty} b^i = \frac{1 - b^{n+1}}{1 - b}$$

**Generalized Arithmetic Series**

For an arithmetic series, we want a coefficient of $i$ for the $i^{th}$ term. We can achieve this by differentiation:

$$\sum_{i=0}^{n} ix^i = x \frac{d}{dx} \sum_{i=0}^{n} x^i = x \frac{d}{dx} \frac{x^{n+1}}{1-x} = \frac{x(nx^{n+1} - (n+1)x^n + 1)}{(1-x)^2}$$
the RHS gives the correct answer for the sum of the series except at \( x = 1 \) where numerator and denominator vanish. However, since the function is smooth near \( x = 1 \), we can differentiate numerator and denominator (twice) wrt \( x \) using l’Hôpital’s rule:

\[
\lim_{{x \to 1}} \sum_{{i=0}}^{{n}} ix^i = \lim_{{x \to 1}} \frac{{(n+2)nx^{n+1}-(n+1)^2x^n+1}}{2(1-x)} \\
= \lim_{{x \to 1}} \frac{{(n+2)(n+1)nx^n-(n+1)^2nx^n-1}}{2} \\
= \frac{{n(n+1)}}{2}
\]

which is the usual formula for the sum of an arithmetic series.

**Negative Binomial Series**

For this series we consider the expansion of

\[
\frac{1}{(1-x)^k} = (1 + x + x^2 + \cdots)^k
\]

To find the coefficient of \( x^m \) in the product, we look at the set of terms which contribute to it, one from each of the \( k \) terms in the RHS. Every such product has coefficient 1, so it is enough to count the number of such combinations. Any such combination \( x^{p_1}x^{p_2} \cdots x^{p_k} \) is a sequence of \( k \) powers \( (p_1, \ldots, p_k) \) which are non-negative integers whose sum is \( m \). This is the number of ordered partitions of \( m \). It has the value \( \binom{m+k-1}{k-1} \). So

\[
\frac{1}{(1-x)^k} = \sum_{{i=0}}^{{\infty}} \binom{m+k-1}{k-1} x^m
\]