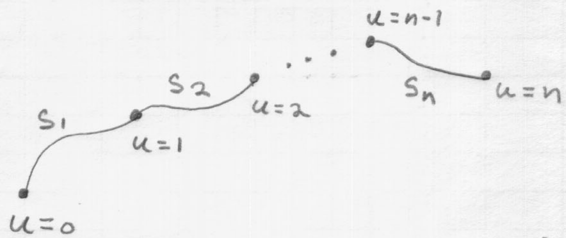
Natural Cubic Splines -- Not local

C^2, interpolate
pw cubic, global



$$x(u) = \begin{cases} S_1(u) & \text{if } 0 \le u < 1 \\ S_2(u) & \text{if } 1 \le u \le 2 \\ \vdots & \\ S_n(u) & \text{if } (n-1) \le u \le n \end{cases}$$

$S_i(1) = p_i \quad i = 1..n \leftarrow n$ constraints
$S_i(0) = p_{i-1} \quad i = 1..n \leftarrow n$ constraints

$S_i'(1) = S_{i+1}'(0) \quad i = 1..n-1$
$S_i''(1) = S_{i+1}''(0) \quad i = 1..n-1$ Another $2n-2$ constraints

$S_1''(0) = S_n''(1) = 0 \longleftarrow$ 2 more constraints

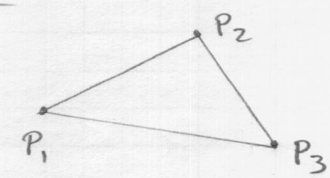
$\rightarrow 4n$ total constraints

Cuck into your favorite linerar alg. package

PS: This scheme interpolates the p_i

? Could you have a G^1 curve the both
- interpolates & has convex hull?
- No!

Try it

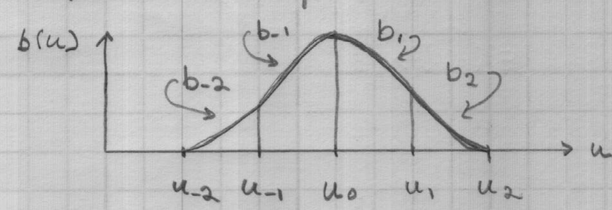


Could do higer order curve if we wished...

ⓘ Can we build C^2 cubic scheme that is local?

B-Splines -- Algebraic construction

Build a hump:



the u_i are "knot" points

$$b(u) = \begin{cases} b_{-2}(u) & \text{if } u \in [u_{-2}, u_{-1}) \\ b_{-1}(u) & u \in [u_{-1}, u_0) \\ b_1(u) & u \in [u_0, u_1) \\ b_2(u) & u \in [u_1, u_2] \\ 0 & \text{else} \end{cases}$$

$$\left.\begin{aligned} b''_{-2}(u_{-2}) = b'_{-2}(u_{-2}) = b_{-2}(u_{-2}) = 0 \\ b''_2(u_2) = b'_2(u_2) = b_2(u_2) = 0 \end{aligned}\right] \quad 6$$

$$\left.\begin{aligned} b_{-2}(u_{-1}) &= b_{-1}(u_{-1}) \\ b_{-1}(u_0) &= b_1(u_0) \\ b_1(u_1) &= b_2(u_1) \end{aligned}\right] \quad \text{repeat for } b' \,\&\, b'' \quad 3 \times 3 = 9$$

$6 + 9 = 15$ need 1 more

Convex hull $\Rightarrow b_{-2}(u_{-2}) + b_{-1}(u_{-1}) + b_1(u_0) + b_2(u_1) = 1$

last constraint gives 16 total

Build curve w/ overlapping bumps

ⓘ What to do w/ ends?

1) Circular

2) repeat first & last ×4

3) add extras

B-Splines -- Geometric Construction
(Cox-de Boor)

Let "hump" centered @ u_i be called $N_{i,4}(u)$

Cubic is Order 4

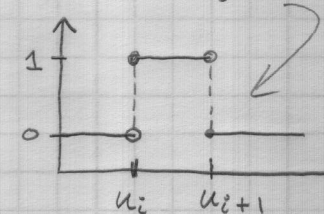
eg

$N_{i,k}(u)$ is Order k B-spline centered @ u_i

Note: i is int if k even
else $(i+1/2)$ is int

[* This notation is different (better) than what most books use.

$N_{i+1/2,1}(u)$



$$N_{i,1}(u) = \begin{cases} 1 & \text{if } u_{i-1/2} \le u < u_{i+1/2} \\ 0 & \text{else} \end{cases}$$

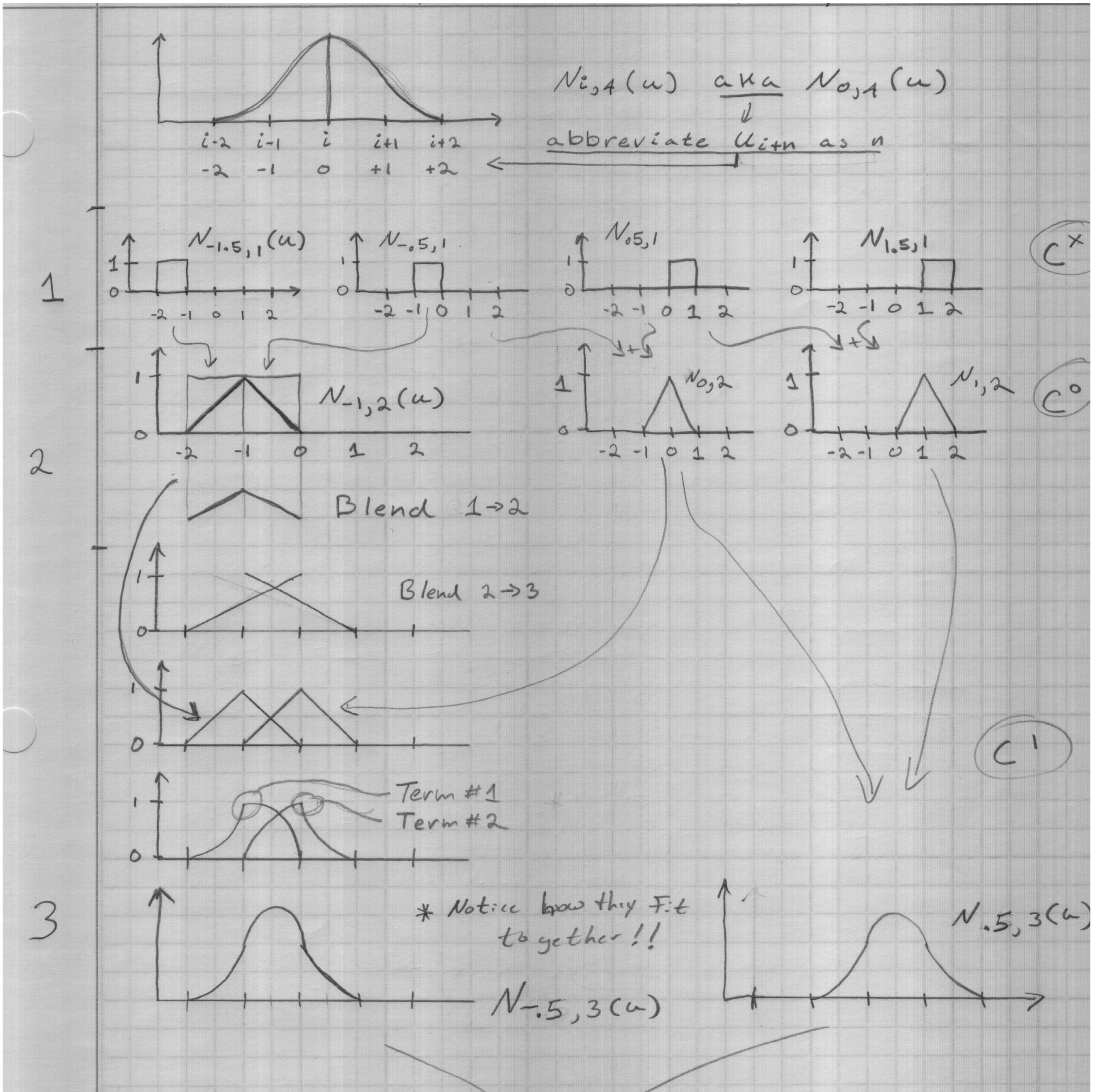
$$N_{i,k}(u) = \frac{(u - u_{i-k/2})\, N_{i-1/2,k-1}(u)}{u_{i+k/2-1} - u_{i-k/2}} + \frac{(u_{i+k/2} - u)\, N_{i+1/2,k-1}(u)}{u_{i+k/2} - u_{i-k/2+1}}$$

$k \ge 2$

"Term #1"

"Term #2"

$\boxed{\text{recursive defn.}}$



4 $N_{0,4}(u)$ C^2

* Too hard to draw...

* Easy to keep going ie $N_{x,\underline{\underline{10}}}(u)$ if you like

* $N_{x,k}$ are C^{k-2}

* What if $u_i = u_{i+1}$ for some i?

NURBS

Non-Uniform Rational B-Splines

* Non-uniform: $u_i - u_{i-1} \neq u_j - u_{j-1}$

  (we've already been assuming this)

* Rational:

  Let $p_i \in \mathbb{R}^{n+1}$ ie $p_i \in \mathbb{R}^4$ if curve in $\mathbb{R}^3$

$$p_i = \begin{bmatrix} p_{ix} \\ p_{iy} \\ p_{iz} \\ p_{iw} \end{bmatrix}$$ ← Homog. Coords.

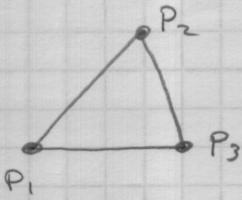
$$x(u) = \frac{\sum_i \begin{bmatrix} p_{ix} \\ p_{ix} \\ p_{iz} \end{bmatrix} N_i(u)}{\sum_i p_{iw} N_i(u)}$$

* Non-linear in P_i

  but Inv. under prj xF.

  ? why so nice?

Barycentric Coords



$$x(u, v) = x(b_1, b_2, b_3) = \sum_{i=1}^{3} b_i P_i$$

$$\sum b_i = 1 \leftarrow b \in R^2$$

$$\forall b_i \geq 0 \Rightarrow \text{inside triangle}$$

Work in R^n w/ appro. simplex (ie $n+1$ points)

↑ object, not space

Basis

$$\underbrace{\begin{bmatrix} u \\ v \\ 1 \end{bmatrix}}_{\bar{u}} = \underbrace{\begin{bmatrix} P_{1x} & P_{2x} & P_{3x} \\ P_{1y} & P_{2y} & P_{3y} \\ 1 & 1 & 1 \end{bmatrix}}_{\beta^{-1}} \cdot \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}}_{b}$$

$$\bar{u} = \beta^{-1} b \Rightarrow b = \beta \bar{u}$$

$$\circledast \longrightarrow x(u) = [P_1 \; P_2 \; P_3] \cdot \beta \cdot \bar{u}$$

Blossum

Arc Length Re-Param

$$x(u) = z(\ell) \quad \text{w/} \quad \int_a^b \| z'(\ell) \| \, d\ell = |b - a|$$