Announcements

- **Optional midterm**
  - On Tuesday 11/21 in class
  - We will count the midterms and final as 1 / 1 / 2, and drop the lowest (or halve the final weight)
  - Will also run grades as if this midterm did not happen
  - You will get the better of the two grades

- **Projects**
  - 3.2 up, do written questions before programming
Recap: Value of Information

- Current evidence \( E=e \), utility depends on \( S=s \)
  
  \[
  \text{MEU}(e) = \max_a \sum_s P(s|e) \ U(s,a)
  \]

- Potential new evidence \( E' \): suppose we knew \( E' = e' \)

  \[
  \text{MEU}(e, e') = \max_a \sum_s P(s|e, e') \ U(s,a)
  \]

- \( \text{BUT} \) \( E' \) is a random variable whose value is currently unknown, so:
  - Must compute expected gain over all possible values

  \[
  \text{VPI}_e(E') = \sum_{e'} P(e'|e) \left( \text{MEU}(e, e') - \text{MEU}(e) \right)
  \]

- \( \text{(VPI = value of perfect information)} \)

VPI Scenarios

- Imagine actions 1 and 2, for which \( U_1 > U_2 \)
- How much will information about \( E_j \) be worth?

Little – we’re sure action 1 is better.

A lot – either could be much better

Little – info likely to change our action but not our utility
Reasoning over Time

- Often, we want to reason about a sequence of observations
  - Speech recognition
  - Robot localization
  - User attention
  - Medical monitoring

- Need to introduce time into our models
- Basic approach: hidden Markov models (HMMs)
- More general: dynamic Bayes’ nets

Markov Models

- A Markov model is a chain-structured BN
  - Each node is identically distributed (stationarity)
  - Value of X at a given time is called the state
  - As a BN:

```
X_1 -> X_2 -> X_3 -> X_4 ...
```

\[ P(X_1), P(X|X_{-1}) \]

- Parameters: called transition probabilities or dynamics, specify how the state evolves over time (also, initial probs)
Conditional Independence

- Basic conditional independence:
  - Past and future independent of the present
  - Each time step only depends on the previous
  - This is called the (first order) Markov property

- Note that the chain is just a (growing BN)
  - We can always use generic BN reasoning on it (if we truncate the chain)

Example: Markov Chain

- Weather:
  - States: \( X = \{ \text{rain, sun} \} \)
  - Transitions:

- Initial distribution: 1.0 sun
- What's the probability distribution after one step?

\[
P(X_2 = \text{sun}) = P(X_2 = \text{sun}|X_1 = \text{sun})P(X_1 = \text{sun}) + P(X_2 = \text{sun}|X_1 = \text{rain})P(X_1 = \text{rain})
\]

\[
0.9 \cdot 1.0 + 0.1 \cdot 0.0 = 0.9
\]
Mini-Forward Algorithm

- Question: probability of being in state $x$ at time $t$?
- Slow answer:
  - Enumerate all sequences of length $t$ which end in $s$
  - Add up their probabilities

\[ P(X_t = \text{sun}) = \sum_{x_1 \ldots x_{t-1}} P(x_1, \ldots, x_{t-1}, \text{sun}) \]

\[
P(X_1 = \text{sun})P(X_2 = \text{sun}|X_1 = \text{sun})P(X_3 = \text{sun}|X_2 = \text{sun})P(X_4 = \text{sun}|X_3 = \text{sun})
\]
\[
P(X_1 = \text{sun})P(X_2 = \text{rain}|X_1 = \text{sun})P(X_3 = \text{sun}|X_2 = \text{rain})P(X_4 = \text{sun}|X_3 = \text{sun})
\]
\[
\vdots
\]

Mini-Forward Algorithm

- Better way: cached incremental belief updates

\[
P(x_t) = \sum_{x_{t-1}} P(x_t|x_{t-1})P(x_{t-1})
\]

\[
P(x_1) = \text{known}
\]

Forward simulation
Example

- From initial observation of sun

\[
\begin{pmatrix}
1.0 \\
0.0
\end{pmatrix}
\begin{pmatrix}
0.9 \\
0.1
\end{pmatrix}
\begin{pmatrix}
0.82 \\
0.18
\end{pmatrix}
\rightarrow
\begin{pmatrix}
0.5 \\
0.5
\end{pmatrix}
\]

\[P(X_1) \quad P(X_2) \quad P(X_3) \quad P(X_\infty)\]

- From initial observation of rain

\[
\begin{pmatrix}
0.0 \\
1.0
\end{pmatrix}
\begin{pmatrix}
0.1 \\
0.9
\end{pmatrix}
\begin{pmatrix}
0.18 \\
0.82
\end{pmatrix}
\rightarrow
\begin{pmatrix}
0.5 \\
0.5
\end{pmatrix}
\]

\[P(X_1) \quad P(X_2) \quad P(X_3) \quad P(X_\infty)\]

Stationary Distributions

- If we simulate the chain long enough:
  - What happens?
  - Uncertainty accumulates
  - Eventually, we have no idea what the state is!

- Stationary distributions:
  - For most chains, the distribution we end up in is independent of the initial distribution
  - Called the stationary distribution of the chain
  - Usually, can only predict a short time out
Web Link Analysis

- PageRank over a web graph
  - Each web page is a state
  - Initial distribution: uniform over pages
  - Transitions:
    - With prob. \( c \), uniform jump to a random page (dotted lines)
    - With prob. \( 1-c \), follow a random outlink (solid lines)

- Stationary distribution
  - Will spend more time on highly reachable pages
  - E.g. many ways to get to the Acrobat Reader download page!
  - Somewhat robust to link spam
  - Google 1.0 returned the set of pages containing all your keywords in decreasing rank, now all search engines use link analysis along with many other factors

Most Likely Explanation

- Question: most likely sequence ending in \( x \) at \( t \)?
  - E.g. if sun on day 4, what’s the most likely sequence?
  - Intuitively: probably sun all four days

- Slow answer: enumerate and score

\[
P(X_t = \text{sun}) = \max_{x_1, \ldots, x_{t-1}} P(x_1, \ldots, x_{t-1}, \text{sun})
\]
\[
P(X_1 = \text{sun})P(X_2 = \text{sun} | X_1 = \text{sun})P(X_3 = \text{sun} | X_2 = \text{sun})P(X_4 = \text{sun} | X_3 = \text{sun})
\]
\[
P(X_1 = \text{sun})P(X_2 = \text{rain} | X_1 = \text{sun})P(X_3 = \text{sun} | X_2 = \text{rain})P(X_4 = \text{sun} | X_3 = \text{sun})
\]
\[
\vdots
\]
Mini-Viterbi Algorithm

- Better answer: cached incremental updates

- Define: \( m_t[x] = \max_{x_{1:t-1}} P(x_{1:t-1}, x) \)
  
  \( a_t[x] = \arg \max_{x_{1:t-1}} P(x_{1:t-1}, x) \)

- Read best sequence off of m and a vectors

---

**Mini-Viterbi**

\[
m_t[x] = \max_{x_{1:t-1}} P(x_{1:t-1}, x)
\]

\[
= \max_{x_{1:t-1}} P(x_{1:t-1}) P(x_{t-1})
\]

\[
= \max_{x_{t-1}} P(x_t|x_{t-1}) \max_{x_{1:t-2}} P(x_{1:t-1})
\]

\[
= \max_{x_{t-1}} P(x_t|x_{t-1}) m_{t-1}[x]
\]

\[
m_1[x] = P(x_1)
\]
Example

\[
\begin{array}{c|c}
 s & \pi(s) \\
\hline
<s> & 1 \\
\end{array}
\]

\[
\begin{array}{c|c|c}
 s' & s & t(s \mid s') \\
\hline
\text{rain} & \text{rain} & 3/5 \\
\text{rain} & \text{sun} & 2/5 \\
\text{sun} & \text{rain} & 1/5 \\
\text{sun} & \text{sun} & 4/5 \\
<s> & \text{rain} & 1/3 \\
<s> & \text{sun} & 2/3 \\
\end{array}
\]

Example

\[
\begin{array}{c|c}
 s' & s \\
\hline
\text{rain} & \text{rain} & 3/5 \\
\text{rain} & \text{sun} & 2/5 \\
\text{sun} & \text{rain} & 1/5 \\
\text{sun} & \text{sun} & 4/5 \\
<s> & \text{rain} & 1/3 \\
<s> & \text{sun} & 2/3 \\
\end{array}
\]
Example

Example
Hidden Markov Models

- Markov chains not so useful for most agents
  - Eventually you don’t know anything anymore
  - Need observations to update your beliefs

- Hidden Markov models (HMMs)
  - Underlying Markov chain over states $S$
  - You observe outputs (effects) at each time step
  - As a Bayes’ net:

$$
\begin{array}{c}
X_1 \rightarrow X_2 \rightarrow X_3 \rightarrow X_4 \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
E_1 \quad E_2 \quad E_3 \quad E_4
\end{array}
$$

Example

- An HMM is
  - Initial distribution: $P(X_1)$
  - Transitions: $P(X|X_{-1})$
  - Emissions: $P(E|X)$
Conditional Independence

- HMMs have two important independence properties:
  - Markov hidden process, future depends on past via the present
  - Current observation independent of all else given current state

- Quiz: does this mean that observations are independent given no evidence?
  - [No, correlated by the hidden state]

Forward Algorithm

- Can ask the same questions for HMMs as Markov chains
- Given current belief state, how to update with evidence?
  - This is called monitoring or filtering
- Formally, we want: \( P(X_t = x_t | e_{1:t}) \)

\[
P(x_t | e_{1:t}) \propto P(x_t, e_{1:t})
= \sum_{x_{t-1}} P(x_{t-1}, x_t, e_{1:t})
= \sum_{x_{t-1}} P(x_{t-1}, e_{1:t-1}) P(x_t | x_{t-1}) P(e_t | x_t)
= P(e_t | x_t) \sum_{x_{t-1}} P(x_t | x_{t-1}) P(x_{t-1}, e_{1:t-1})
\]
Example

\[ P(x_t|e_{1:t}) \propto f_t[x_t] = P(x_t, e_{1:t}) \]

\[ f_t[x_t] = P(e_t|x_t) \sum_{x_{t-1}} P(x_t|x_{t-1}) f_{t-1}[x_{t-1}] \]

Viterbi Algorithm

- **Question:** what is the most likely state sequence given the observations?
  - **Slow answer:** enumerate all possibilities
  - **Better answer:** cached incremental version

\[ x^*_{1:T} = \arg \max_{x_{1:T}} P(x_{1:T}|e_{1:T}) \]

\[ m_t[x_t] = \max_{x_{1:t-1}} P(x_{1:t-1}, x_t, e_{1:t}) \]

\[ = \max_{x_{1:t-1}} P(x_{1:t-1}, e_{1:t-1}) P(x_t|x_{t-1}) P(e_t|x_t) \]

\[ = P(e_t|x_t) \max_{x_{t-1}} P(x_t|x_{t-1}) \max_{x_{1:t-2}} P(x_{1:t-1}, e_{1:t-1}) \]

\[ = P(e_t|x_t) \max_{x_{t-1}} P(x_t|x_{t-1}) m_{t-1}[x_{t-1}] \]
Example

Real HMM Examples

- **Speech recognition HMMs:**
  - Observations are acoustic signals (continuous valued)
  - States are specific positions in specific words (so, tens of thousands)

- **Machine translation HMMs:**
  - Observations are words (tens of thousands)
  - States are translation positions (dozens)

- **Robot tracking:**
  - Observations are range readings (continuous)
  - States are positions on a map (continuous)