Q1. Conditional Independence

The figures below show pairs of Bayes Nets. In each, the ORIGINAL network is shown on the left. The REVERSED network, shown on the right, has had all the arrows flipped. Therefore, the REVERSED network may not be able to represent the ORIGINAL distribution. For each pair, add a minimal number of arrows to the REVERSED network such that it is guaranteed to be able to represent the distribution in the ORIGINAL network. If no arrows need to be added, clearly state none needed.

(a) Original: P → Q → R
   Reversed: P → Q → R

(b) Original: P → Q → R
   Reversed: P → Q → R

(c) Original: P → Q → S → R
   Reversed: P → Q → S → R
   OR
(d) Consider the Bayes net structure below, which encodes some probability distribution of the form $P(W, X, Y, Z)$. For some specific value of $W = w$, let $Q(X, Y, Z) = P(X, Y, Z|w)$ be the corresponding posterior distribution. On the right, draw the smallest (fewest arcs) Bayes net that is guaranteed to be able to represent the distribution $Q$. 

\begin{center}
\begin{tikzpicture}
  \node[circle,draw] (P) at (0,2) {$P$};
  \node[circle,draw] (Q) at (1,2) {$Q$};
  \node[circle,draw] (S) at (0,1) {$S$};
  \node[circle,draw] (R) at (1,1) {$R$};
  \draw[->] (P) -- (S);
  \draw[->] (P) -- (R);
  \draw[->] (Q) -- (S);
  \draw[->] (Q) -- (R);
  \node at (-.5,0) {ORIGINAL NET};

  \begin{scope}[xshift=3cm]
    \node[circle,draw] (P) at (0,1) {$P$};
    \node[circle,draw] (Q) at (1,1) {$Q$};
    \node[circle,draw] (S) at (0,0) {$S$};
    \node[circle,draw] (R) at (1,0) {$R$};
    \draw[->,red] (S) -- (R);
    \draw[->,red] (R) -- (S);
    \node at (-.5,0) {REVERSED: ADD ARROWS};
  \end{scope}
\end{tikzpicture}
\end{center}

\begin{center}
\begin{tikzpicture}
  \node[circle,draw] (W) at (0,0) {$W$};
  \node[circle,draw] (X) at (-1,1) {$X$};
  \node[circle,draw] (Y) at (1,1) {$Y$};
  \node[circle,draw] (Z) at (0,-1) {$Z$};
  \draw[->] (W) -- (X);
  \draw[->] (W) -- (Y);
  \node at (-.5,-2) {P(X, Y, Z, W)};

  \begin{scope}[xshift=3cm]
    \node[circle,draw] (X) at (0,0) {$X$};
    \node[circle,draw] (Y) at (0,1) {$Y$};
    \node[circle,draw] (Z) at (0,-2) {$Z$};
    \draw[->] (X) -- (Y);
    \node at (-.5,-2) {Q(X, Y, Z)};
  \end{scope}
\end{tikzpicture}
\end{center}
Q2. Basic Inference

The next parts involve computing various quantities in the network below. These questions are designed so that they can be answered with a minimum of computation. If you find yourself doing a copious amount of computation for each part, step back and consider whether there is simpler way to deduce the answer.

(a) $P(a, \neg b, c, \neg d)$

$P(a)P(\neg b|a)P(c|a)P(\neg d|\neg b) = 0.1 \times 0.5 \times 0.4 \times 0.8 = 0.016$

(b) $P(b)$

$P(b) = \sum_{A=\{a, \neg a\}} P(A)P(b|A) = 0.1 \times 0.5 + 0.9 \times 0.8 = 0.77$

(c) $P(a|b)$

$P(a|b) = \frac{P(a,b)}{P(b)} = \frac{P(a)P(b|a)}{P(b)} = \frac{0.1 \times 0.5}{0.77} = 0.064935$

(d) $P(d|a)$

$P(d|a) = \sum_{B=\{b, \neg b\}} P(d|B)p(B|a) = 0.9 \times 0.5 + 0.2 \times 0.5 = 0.55$

(e) $P(d|a,c)$

From the conditional independence properties of the graph, $D \indep C|\{A\}$. Hence, $P(d|a,c) = p(d|a) = 0.55$