CS 188 Fall 2018 | Introduction to Artificial Intelligence

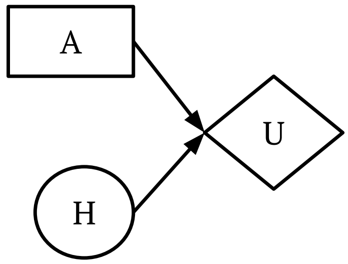
Written HW 8 Sol.

Self-assessment due: Monday 11/5/2018 at 11:59pm (submit via Gradescope)

For the self assessment, **fill in the self assessment boxes in your original submission** (you can download a PDF copy of your submission from Gradescope – be sure to delete any extra title pages that Gradescope attaches). For each subpart where your original answer was correct, write "correct." Otherwise, write and explain the correct answer. **Do not leave any boxes empty.** If you did not submit the homework (or skipped some questions) but wish to receive credit for the self-assessment, we ask that you first complete the homework without looking at the solutions, and then perform the self assessment afterwards.

Q1. Decision Networks

After years of battles between the ghosts and Pacman, the ghosts challenge Pacman to a winner-take-all showdown, and the game is a coin flip. Pacman has a decision to make: whether to accept the challenge (*accept*) or decline (*decline*). If the coin comes out heads ($+h$) Pacman wins. If the coin comes out tails ($-h$), the ghosts win. No matter what decision Pacman makes, the outcome of the coin is revealed.



H	$P(H)$
+h	0.5
-h	0.5

H	A	U(H,A)
+h	*accept*	100
-h	*accept*	-100
+h	*decline*	-30
-h	*decline*	50

(a) Maximum Expected Utility

Compute the following quantities:

$EU(accept) = P(+h)U(+h, accept) + P(-h)U(-h, accept) = 0.5 * 100 + 0.5 * -100 = 0$

$EU(decline) = P(+h)U(+h, decline) + P(-h)U(-h, decline) = 0.5 * -30 + 0.5 * 50 = 10$

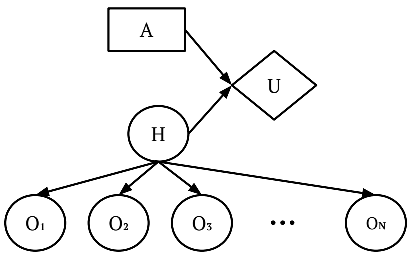
$MEU(\{\}) = max(0, 10) = 10$

Action that achieves $MEU(\{\})$ = *decline*

(b) VPI relationships When deciding whether to accept the winner-take-all coin flip, Pacman can consult a few fortune tellers that he knows. There are N fortune tellers, and each one provides a prediction O_n for H.

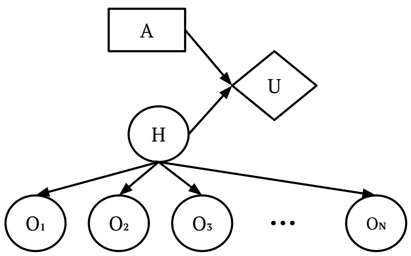
For each of the questions below, select **all** of the VPI relations that are guaranteed to be true, or select *None of the above.*

(i) In this situation, the fortune tellers give perfect predictions.
Specifically, $P(O_n = +h \mid H = +h) = 1$, $P(O_n = -h \mid H = -h) = 1$, for all n from 1 to N.



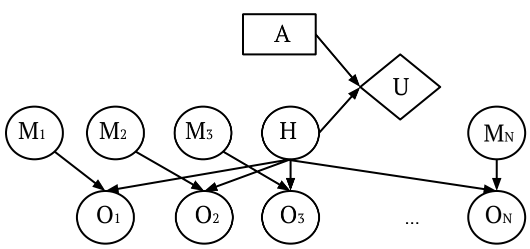
- ☐ $\text{VPI}(O_1,\ O_2) \geq \text{VPI}(O_1) + \text{VPI}(O_2)$
- ■ $\text{VPI}(O_i) = \text{VPI}(O_j)$ where $i \neq j$
- ☐ $\text{VPI}(O_3 \mid O_2,\ O_1) > \text{VPI}(O_2 \mid O_1)$.
- ☐ $\text{VPI}(H) > \text{VPI}(O_1, O_2, \ldots O_N)$
- ☐ None of the above.

(ii) In another situation, the fortune tellers are pretty good, but not perfect.
Specifically, $P(O_n = +h \mid H = +h) = 0.8$, $P(O_n = -h \mid H = -h) = 0.5$, for all n from 1 to N.



- ☐ $\text{VPI}(O_1,\ O_2) \geq \text{VPI}(O_1) + \text{VPI}(O_2)$
- ■ $\text{VPI}(O_i) = \text{VPI}(O_j)$ where $i \neq j$
- ☐ $\text{VPI}(O_3 \mid O_2,\ O_1) > \text{VPI}(O_2 \mid O_1)$.
- ■ $\text{VPI}(H) > \text{VPI}(O_1, O_2, \ldots O_N)$
- ☐ None of the above.

(iii) In a third situation, each fortune teller's prediction is affected by their mood. If the fortune teller is in a good mood ($+m$), then that fortune teller's prediction is guaranteed to be correct. If the fortune teller is in a bad mood ($-m$), then that teller's prediction is guaranteed to be incorrect. Each fortune teller is happy with probability $P(M_n = +m) = 0.8$.

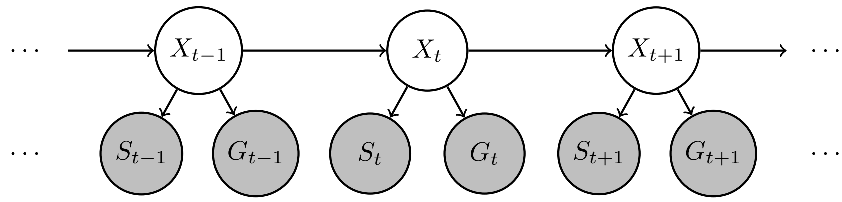


- ☐ $\text{VPI}(M_1) > 0$
- ■ $\forall i\ \ \text{VPI}(M_i | O_i) > 0$
- ☐ $\text{VPI}(M_1,\ M_2,\ \ldots,\ M_N) > \text{VPI}(M_1)$
- ■ $\forall i\ \ \text{VPI}(H) = \text{VPI}(M_i, O_i)$
- ☐ None of the above.

Q2. HMM: Where is the Car?

Transportation researchers are trying to improve traffic in the city but, in order to do that, they first need to estimate the location of each of the cars in the city. They need our help to model this problem as an inference problem of an HMM. For this question, assume that only *one* car is being modeled.

(a) The structure of this modified HMM is given below, which includes X, the location of the car; S, the noisy location of the car from the signal strength at a nearby cell phone tower; and G, the noisy location of the car from GPS.



We want to perform filtering with this HMM. That is, we want to compute the belief $P(x_t|s_{1:t}, g_{1:t})$, the probability of a state x_t given all past and current observations.

The **dynamics update** expression has the following form:

$$P(x_t|s_{1:t-1}, g_{1:t-1}) = \underline{\quad\textbf{(i)}\quad} \; \underline{\quad\textbf{(ii)}\quad} \; \underline{\quad\textbf{(iii)}\quad} \; P(x_{t-1}|s_{1:t-1}, g_{1:t-1}).$$

Complete the expression by choosing the option that fills in each blank.

(i)	○ $P(s_{1:t}, g_{1:t})$	○ $P(s_{1:t-1}, g_{1:t-1})$	○ $P(s_{1:t-1})P(g_{1:t-1})$	○ $P(s_{1:t})P(g_{1:t})$	● 1
(ii)	○ $\sum_{x_t}$	● $\sum_{x_{t-1}}$	○ $\max_{x_{t-1}}$	○ $\max_{x_t}$	○ 1
(iii)	○ $P(x_{t-1} \mid x_{t-2})$	○ $P(x_{t-2}, x_{t-1})$	○ $P(x_{t-1}, x_t)$	● $P(x_t|x_{t-1})$	○ 1

The derivation of the dynamics update is similar to the one for the canonical HMM, but with two observation variables instead.

$$\begin{aligned} P(x_t|s_{1:t-1}, g_{1:t-1}) &= \sum_{x_{t-1}} P(x_{t-1}, x_t|s_{1:t-1}, g_{1:t-1}) \\ &= \sum_{x_{t-1}} P(x_t|x_{t-1}, s_{1:t-1}, g_{1:t-1})P(x_{t-1}|s_{1:t-1}, g_{1:t-1}) \\ &= \sum_{x_{t-1}} P(x_t|x_{t-1})P(x_{t-1}, x_t|s_{1:t-1}, g_{1:t-1}) \end{aligned}$$

In the last step, we use the independence assumption given in the HMM, $X_t \perp\!\!\!\perp S_{1:t-1}, G_{1:t-1}|X_{t-1}$.

The **observation update** expression has the following form:

$$P(x_t|s_{1:t}, g_{1:t}) = \underline{\quad\textbf{(iv)}\quad}\ \underline{\quad\textbf{(v)}\quad}\ \underline{\quad\textbf{(vi)}\quad}\ P(x_t|s_{1:t-1}, g_{1:t-1}).$$

Complete the expression by choosing the option that fills in each blank.

(iv)
- ○ $P(s_{1:t-1}|s_t)P(g_{1:t-1}|g_t)$
- ● $\frac{1}{P(s_t, g_t|s_{1:t-1}, g_{1:t-1})}$
- ○ $\frac{1}{P(s_{1:t-1}, g_{1:t-1}|s_t, g_t)}$
- ○ $P(s_t, g_t|s_{1:t-1}, g_{1:t-1})$
- ○ $P(s_{1:t-1}, g_{1:t-1}|s_t, g_t)$
- ○ $P(s_t|s_{1:t-1})P(g_t|g_{1:t-1})$
- ○ $\frac{1}{P(s_t|s_{1:t-1})P(g_t|g_{1:t-1})}$
- ○ $\frac{1}{P(s_{1:t-1}|s_t)P(g_{1:t-1}|g_t)}$
- ○ 1

(v)
- ○ $\sum_{x_t}$
- ○ $\sum_{x_{t-1}}$
- ○ $\max_{x_t}$
- ○ $\max_{x_{t-1}}$
- ● 1

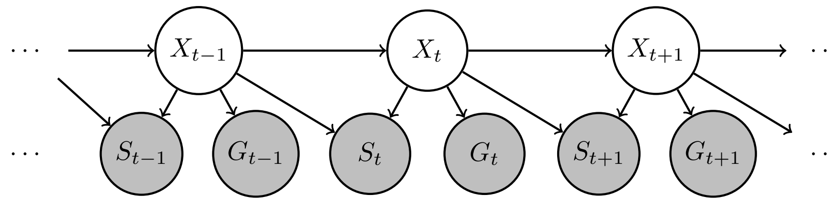
(vi)
- ○ $P(x_{t-1}, s_{t-1})P(x_{t-1}, g_{t-1})$
- ○ $P(x_{t-1}, s_{t-1}, g_{t-1})$
- ○ $P(x_t|s_t)P(x_t|g_t)$
- ○ $P(s_{t-1}|x_{t-1})P(g_{t-1}|x_{t-1})$
- ○ $P(x_t, s_t)P(x_t, g_t)$
- ○ $P(x_t, s_t, g_t)$
- ○ $P(x_{t-1}|s_{t-1})P(x_{t-1}|g_{t-1})$
- ● $P(s_t|x_t)P(g_t|x_t)$
- ○ 1

Again, the derivation of the observation update is similar to the one for the canonical HMM, but with two observation variables instead.

$$\begin{aligned}
P(x_t|s_{1:t}, g_{1:t}) &= P(x_t|s_t, g_t, s_{1:t-1}, g_{1:t-1}) \\
&= \frac{1}{P(s_t, g_t|s_{1:t-1}, g_{1:t-1})} P(x_t, s_t, g_t|s_{1:t-1}, g_{1:t-1}) \\
&= \frac{1}{P(s_t, g_t|s_{1:t-1}, g_{1:t-1})} P(s_t, g_t|x_t, s_{1:t-1}, g_{1:t-1}) P(x_t|s_{1:t-1}, g_{1:t-1}) \\
&= \frac{1}{P(s_t, g_t|s_{1:t-1}, g_{1:t-1})} P(s_t, g_t|x_t) P(x_t|s_{1:t-1}, g_{1:t-1}) \\
&= \frac{1}{P(s_t, g_t|s_{1:t-1}, g_{1:t-1})} P(s_t|x_t) P(g_t|x_t) P(x_t|s_{1:t-1}, g_{1:t-1})
\end{aligned}$$

In the second to last step, we use the independence assumption $S_t, G_t \perp\!\!\!\perp S_{1:t-1}, G_{1:t-1}|X_t$; and in the last step, we use the independence assumption $S_t \perp\!\!\!\perp G_t|X_t$.

(b) It turns out that if the car moves too fast, the quality of the cell phone signal decreases. Thus, the signal-dependent location S_t not only depends on the current state X_t but it also depends on the previous state X_{t-1}. Thus, we modify our original HMM for a new more accurate one, which is given below.



Again, we want to compute the belief $P(x_t|s_{1:t}, g_{1:t})$. In this part we consider an update that combines the dynamics and observation update in a *single* update.

$$P(x_t|s_{1:t}, g_{1:t}) = \underline{\quad\textbf{(i)}\quad}\ \underline{\quad\textbf{(ii)}\quad}\ \underline{\quad\textbf{(iii)}\quad}\ \underline{\quad\textbf{(iv)}\quad}\ P(x_{t-1}|s_{1:t-1}, g_{1:t-1}).$$

Complete the **forward update** expression by choosing the option that fills in each blank.

(i)
- ○ $P(s_{1:t-1}, g_{1:t-1}|s_t, g_t)$
- ○ $P(s_t, g_t|s_{1:t-1}, g_{1:t-1})$
- ○ $P(s_t|s_{1:t-1})P(g_t|g_{1:t-1})$
- ● $\frac{1}{P(s_t, g_t|s_{1:t-1}, g_{1:t-1})}$
- ○ $\frac{1}{P(s_{1:t-1}, g_{1:t-1}|s_t, g_t)}$
- ○ $P(s_{1:t-1}|s_t)P(g_{1:t-1}|g_t)$
- ○ $\frac{1}{P(s_t|s_{1:t-1})P(g_t|g_{1:t-1})}$
- ○ $\frac{1}{P(s_{1:t-1}|s_t)P(g_{1:t-1}|g_t)}$
- ○ 1

(ii)
- ○ $\max_{x_{t-1}}$
- ○ $\max_{x_t}$
- ● $\sum_{x_{t-1}}$
- ○ $\sum_{x_t}$
- ○ 1

(iii)
- ○ $P(s_{t-1}|x_{t-2}, x_{t-1})P(g_{t-1}|x_{t-1})$
- ● $P(s_t|x_{t-1}, x_t)P(g_t|x_t)$
- ○ $P(s_t, g_t|x_t)$
- ○ $P(x_{t-2}, x_{t-1}, s_{t-1})P(x_{t-1}, g_{t-1})$
- ○ $P(x_{t-1}, x_t, s_t)P(x_t, g_t)$
- ○ $P(s_{t-1}, g_{t-1}|x_{t-1})$
- ○ $P(x_{t-2}, x_{t-1}|s_{t-1})P(x_{t-1}|g_{t-1})$
- ○ $P(x_{t-1}, x_t|s_t)P(x_t|g_t)$
- ○ 1
- ○ $P(x_{t-2}, x_{t-1}, s_{t-1}, g_{t-1})$
- ○ $P(x_{t-1}, x_t, s_t, g_t)$

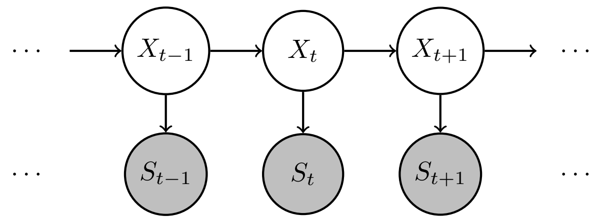
(iv)
- ○ $P(x_{t-1}, x_t)$
- ● $P(x_t|x_{t-1})$
- ○ $P(x_{t-2}, x_{t-1})$
- ○ $P(x_{t-1}|x_{t-2})$
- ○ 1

For this modified HMM, we have the dynamics and observation update in a single update because one of the previous independence assumptions does not longer holds.

$$\begin{aligned}
P(x_t|s_{1:t}, g_{1:t}) &= \sum_{x_{t-1}} P(x_{t-1}, x_t|s_t, g_t, s_{1:t-1}, g_{1:t-1}) \\
&= \frac{1}{P(s_t, g_t|s_{1:t-1}, g_{1:t-1})} \sum_{x_{t-1}} P(x_{t-1}, x_t, s_t, g_t|s_{1:t-1}, g_{1:t-1}) \\
&= \frac{1}{P(s_t, g_t|s_{1:t-1}, g_{1:t-1})} \sum_{x_{t-1}} P(s_t, g_t|x_{t-1}, x_t, s_{1:t-1}, g_{1:t-1})P(x_{t-1}, x_t|s_{1:t-1}, g_{1:t-1}) \\
&= \frac{1}{P(s_t, g_t|s_{1:t-1}, g_{1:t-1})} \sum_{x_{t-1}} P(s_t, g_t|x_{t-1}, x_t)P(x_t|x_{t-1}, s_{1:t-1}, g_{1:t-1})P(x_{t-1}|s_{1:t-1}, g_{1:t-1}) \\
&= \frac{1}{P(s_t, g_t|s_{1:t-1}, g_{1:t-1})} \sum_{x_{t-1}} P(s_t|x_{t-1}, x_t)P(g_t|x_{t-1}, x_t)P(x_t|x_{t-1})P(x_{t-1}|s_{1:t-1}, g_{1:t-1}) \\
&= \frac{1}{P(s_t, g_t|s_{1:t-1}, g_{1:t-1})} \sum_{x_{t-1}} P(s_t|x_{t-1}, x_t)P(g_t|x_t)P(x_t|x_{t-1})P(x_{t-1}|s_{1:t-1}, g_{1:t-1})
\end{aligned}$$

In the third to last step, we use the independence assumption $S_t, G_t \perp\!\!\!\perp S_{1:t-1}, G_{1:t-1}|X_{t-1}, X_t$; in the second to last step, we use the independence assumption $S_t \perp\!\!\!\perp G_t|X_{t-1}, X_t$ and $X_t \perp\!\!\!\perp S_{1:t-1}, G_{1:t-1}|X_{t-1}$; and in the last step, we use the independence assumption $G_t \perp\!\!\!\perp X_{t-1}|X_t$.

(c) The Viterbi algorithm finds the most probable sequence of hidden states $X_{1:T}$, given a sequence of observations $s_{1:T}$, for some time $t = T$. Recall the canonical HMM structure, which is shown below.



For this canonical HMM, the Viterbi algorithm performs the following dynamic programming computations:

$$m_t[x_t] = P(s_t|x_t) \max_{x_{t-1}} P(x_t|x_{t-1}) m_{t-1}[x_{t-1}].$$

We consider extending the Viterbi algorithm for the modified HMM from part **(b)**. We want to find the most likely sequence of states $X_{1:T}$ given the sequence of observations $s_{1:T}$ and $g_{1:T}$. The dynamic programming update for $t > 1$ for the modified HMM has the following form:

$$m_t[x_t] = \underline{\quad \textbf{(i)} \quad} \; \underline{\quad \textbf{(ii)} \quad} \; \underline{\quad \textbf{(iii)} \quad} \; m_{t-1}[x_{t-1}].$$

Complete the expression by choosing the option that fills in each blank.

(i) ○ $\sum_{x_{t-1}}$ ○ $\sum_{x_t}$ ○ $\max_{x_t}$ ● $\max_{x_{t-1}}$ ○ 1

(ii) ○ $P(s_{t-1}|x_{t-2}, x_{t-1})P(g_{t-1}|x_{t-1})$ ● $P(s_t|x_{t-1}, x_t)P(g_t|x_t)$ ○ $P(s_t, g_t|x_t)$
○ $P(x_{t-2}, x_{t-1}, s_{t-1})P(x_{t-1}, g_{t-1})$ ○ $P(x_{t-1}, x_t, s_t)P(x_t, g_t)$ ○ $P(s_{t-1}, g_{t-1}|x_{t-1})$
○ $P(x_{t-2}, x_{t-1}|s_{t-1})P(x_{t-1}|g_{t-1})$ ○ $P(x_{t-1}, x_t|s_t)P(x_t|g_t)$ ○ 1
○ $P(x_{t-2}, x_{t-1}, s_{t-1}, g_{t-1})$ ○ $P(x_{t-1}, x_t, s_t, g_t)$

(iii) ○ $P(x_{t-1}, x_t)$ ● $P(x_t|x_{t-1})$ ○ $P(x_{t-2}, x_{t-1})$ ○ $P(x_{t-1}|x_{t-2})$ ○ 1

If we remove the summation from the forward update equation of part **(b)**, we get a joint probability of the states,

$$P(x_{1:t}|s_{1:t}, g_{1:t}) = \frac{1}{P(s_t, g_t|s_{1:t-1}, g_{1:t-1})} P(s_t|x_{t-1}, x_t)P(g_t|x_t)P(x_t|x_{t-1})P(x_{1:t-1}|s_{1:t-1}, g_{1:t-1}).$$

We can define $m_t[x_t]$ to be the maximum joint probability of the states (for a particular x_t) given all past and current observations, times some constant, and then we can find a recursive relationship for $m_t[x_t]$,

$$\begin{aligned}
m_t[x_t] &= P(s_{1:t}, g_{1:t}) \max_{x_{1:t-1}} P(x_{1:t}|s_{1:t}, g_{1:t}) \\
&= P(s_{1:t}, g_{1:t}) \max_{x_{1:t-1}} \frac{1}{P(s_t, g_t|s_{1:t-1}, g_{1:t-1})} P(s_t|x_{t-1}, x_t)P(g_t|x_t)P(x_t|x_{t-1})P(x_{1:t-1}|s_{1:t-1}, g_{1:t-1}) \\
&= \max_{x_{t-1}} P(s_t|x_{t-1}, x_t)P(g_t|x_t)P(x_t|x_{t-1}) \frac{P(s_{1:t}, g_{1:t})}{P(s_t, g_t|s_{1:t-1}, g_{1:t-1})} \max_{x_{1:t-2}} P(x_{1:t-1}|s_{1:t-1}, g_{1:t-1}) \\
&= \max_{x_{t-1}} P(s_t|x_{t-1}, x_t)P(g_t|x_t)P(x_t|x_{t-1})P(s_{1:t-1}, g_{1:t-1}) \max_{x_{1:t-2}} P(x_{1:t-1}|s_{1:t-1}, g_{1:t-1}) \\
&= \max_{x_{t-1}} P(s_t|x_{t-1}, x_t)P(g_t|x_t)P(x_t|x_{t-1}) m_{t-1}[x_{t-1}].
\end{aligned}$$

Notice that the maximum joint probability of states up to time $t = T$ given all past and current observations is given by

$$\max_{x_{1:T}} P(x_{1:T}|s_{1:T}, g_{1:T}) = \frac{\max_{x_t} m_T[x_t]}{P(s_{1:T}, g_{1:T})}.$$

We can recover the actual most likely sequence of states by bookkepping back pointers of the states the maximized the Viterbi update equations.