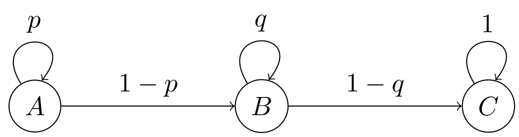
CS188 Fall 2018 Section 9: Midterm 2 Prep

1 . A Not So Random Walk

Pacman is trying to predict the position of a ghost, which he knows has the following transition graph:



Here, $0 < p < 1$ and $0 < q < 1$ are arbitrary probabilities. It is known that the ghost always starts in state A. For this problem, we consider time to begin at 0. For example, at time 0, the ghost is in A with probability 1, and at time 1, the ghost is in A with probability p or in B with probability $1 - p$.

In all of the following questions, you may assume that n is large enough so that the given event occurs with non-zero probability.

(i) Suppose $p \neq q$. What is the probability that the ghost is in A at time n?
For the ghost to be in A at time n, it must have stayed in A for n steps, which occurs with probability

$$p^n.$$

(ii) Suppose $p \neq q$. What is the probability that the ghost first reaches B at time n?
For the ghost to first reach B at time n, it must have stayed in A for $n-1$ steps, then transitioned to B. This occurs with probability

$$f_{(\mathrm{i})}(n-1) \cdot (1-p) = p^{n-1}(1-p).$$

(iii) Suppose $p \neq q$. What is the probability that the ghost is in B at time n?
For the ghost to be in B at time n, it must have first reached B at time i for some $1 \leq i \leq n$, then stayed there for $n - i$ steps. Summing over all values of i gives

$$\sum_{i=1}^{n} f_{(\mathrm{ii})}(i) \cdot q^{n-i} = \sum_{i=1}^{n} p^{i-1}(1-p)q^{n-i} = \frac{(1-p)q^n}{p} \sum_{i=1}^{n} \left(\frac{p}{q}\right)^i = \frac{(1-p)q^n}{p} \cdot \frac{p}{q} \cdot \frac{1-\left(\frac{p}{q}\right)^n}{1-\frac{p}{q}} = (1-p)\frac{q^n - p^n}{q-p}.$$

(iv) Suppose $p \neq q$. What is the probability that the ghost first reaches C at time n?
For the ghost to first reach C at time n, it must have been in B at time $n-1$, then transitioned to C. This occurs with probability

$$f_{(\mathrm{iii})}(n-1) \cdot (1-q) = (1-p)\frac{q^{n-1} - p^{n-1}}{q-p}(1-q).$$

(v) Suppose $p \neq q$. What is the probability that the ghost is in C at time n?
For the ghost to be in C at time n, it must not be in A or B at time n. This occurs with probability

$$1 - f_{(\mathrm{i})}(n) - f_{(\mathrm{iii})}(n) = 1 - p^n - (1-p)\frac{q^n - p^n}{q-p}.$$

Alternatively, for the ghost to be in C at time n, it must have first reached C at time i for some $2 \leq i \leq n$, then stayed there for $n-i$ steps. Note that we can equivalently range over $1 \leq i \leq n$ for computational convenience, since $f_{(\mathrm{iv})}(1) = 0$. Summing over all values of i gives

$$\begin{aligned}
\sum_{i=2}^{n} f_{(\mathrm{iv})}(i) \cdot 1^{n-i} = \sum_{i=1}^{n} f_{(\mathrm{iv})}(i) \cdot 1^{n-i} &= \sum_{i=1}^{n} (1-p)\frac{q^{i-1} - p^{i-1}}{q-p}(1-q) \\
&= \frac{(1-p)(1-q)}{q-p}\left(\frac{1-q^n}{1-q} - \frac{1-p^n}{1-p}\right) = \frac{(1-p)(1-q^n) - (1-q)(1-p^n)}{q-p},
\end{aligned}$$

which is equivalent to the previous expression.

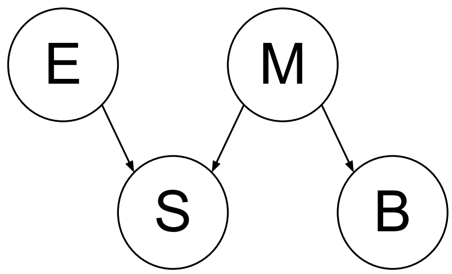
2 . December 21, 2012

A smell of sulphur (S) can be caused either by rotten eggs (E) or as a sign of the doom brought by the Mayan Apocalypse (M). The Mayan Apocalypse also causes the oceans to boil (B). The Bayesian network and corresponding conditional probability tables for this situation are shown below. For each part, you should give either a numerical answer (e.g. 0.81) or an arithmetic expression in terms of numbers from the tables below (e.g. $0.9 \cdot 0.9$).

Note: be careful of doing unnecessary computation here.

$P(E)$	
$+e$	0.4
$-e$	0.6

$P(S \mid E, M)$			
$+e$	$+m$	$+s$	1.0
$+e$	$+m$	$-s$	0.0
$+e$	$-m$	$+s$	0.8
$+e$	$-m$	$-s$	0.2
$-e$	$+m$	$+s$	0.3
$-e$	$+m$	$-s$	0.7
$-e$	$-m$	$+s$	0.1
$-e$	$-m$	$-s$	0.9



$P(M)$	
$+m$	0.1
$-m$	0.9

$P(B \mid M)$		
$+m$	$+b$	1.0
$+m$	$-b$	0.0
$-m$	$+b$	0.1
$-m$	$-b$	0.9

(a) Compute the following entry from the joint distribution:

$P(-e, -s, -m, -b) =$ $P(-e)P(-m)P(-s \mid -e, -m)P(-b \mid -m) = (0.6)(0.9)(0.9)(0.9) = 0.4374$
by expanding the joint according to the chain rule of conditional probability.

(b) What is the probability that the oceans boil?

$P(+b) =$ $P(+b \mid +m)P(+m) + P(+b \mid -m)P(-m) = (1.0)(0.1) + (0.1)(0.9) = 0.19$
by marginalizing out m according to the law of total probability.

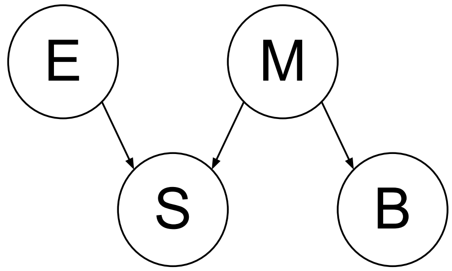
(c) What is the probability that the Mayan Apocalypse is occurring, given that the oceans are boiling?

$P(+m \mid +b) =$ $\frac{P(+b \mid +m)P(+m)}{P(+b)} = \frac{(1.0)(0.1)}{0.19} \approx .5263$
by the definition of conditional probability.

The figures and table below are identical to the ones on the previous page and are repeated here for your convenience.

$P(E)$	
$+e$	0.4
$-e$	0.6

$P(S\|E,M)$			
$+e$	$+m$	$+s$	1.0
$+e$	$+m$	$-s$	0.0
$+e$	$-m$	$+s$	0.8
$+e$	$-m$	$-s$	0.2
$-e$	$+m$	$+s$	0.3
$-e$	$+m$	$-s$	0.7
$-e$	$-m$	$+s$	0.1
$-e$	$-m$	$-s$	0.9



$P(M)$	
$+m$	0.1
$-m$	0.9

$P(B\|M)$		
$+m$	$+b$	1.0
$+m$	$-b$	0.0
$-m$	$+b$	0.1
$-m$	$-b$	0.9

(d) What is the probability that the Mayan Apocalypse is occurring, given that there is a smell of sulphur, the oceans are boiling, and there are rotten eggs?

$P(+m|+s,+b,+e) =$

$$\frac{P(+m,+s,+b,+e)}{\sum_m P(m,+s,+b,+e)} = \frac{P(+e)P(+m)P(+s|+e,+m)P(+b|+m))}{\sum_m P(+e)P(m)P(+s|+e,m)P(+b|m)}$$
$$= \frac{(0.4)(0.1)(1.0)(1.0)}{(0.4)(0.1)(1.0)(1.0)+(0.4)(0.9)(0.8)(0.1)}$$
$$= \frac{0.04}{0.04+0.0288} \approx .5814$$

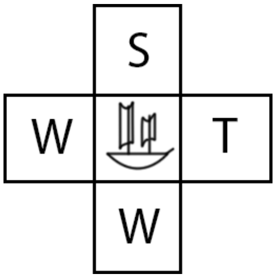
(e) What is the probability that rotten eggs are present, given that the Mayan Apocalypse is occurring?

$P(+e|+m) =$ $P(+e) = 0.4$

The first equality holds true as we have $E \perp\!\!\!\perp M$ (E is independent of M), which can be inferred from the graph of the Bayes' net.

3 . Argg! Sampling for the Legendary Treasure

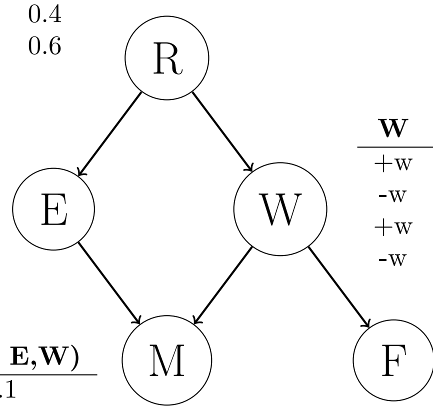
Little did you know that Jasmine and Katie are actually infamous pirates. One day, they go treasure hunting in the Ocean of Bayes, where rumor says a great treasure lies in wait for explorers who dare navigate in the rough waters. After navigating about the ocean, they are within grasp of the treasure. Their current configuration is represented by the boat in the figure below. They can only make one move, and must choose from the actions: (North, South, East, West). Stopping is not allowed. They will land in either a whirlpool (W), an island with a small treasure (S), or an island with the legendary treasure (T). The utilities of the three types of locations are shown below:



State	U(State)
T (Legendary Treasure)	100
S (Small Treasure)	25
W (Whirlpool)	-50

The success of their action depends on the random variable **Movement (M)**, which takes on one of two values: (+m, -m). The Movement random variable has many relationships with other variables: Presence of Enemy Pirates (E), Rain (R), Strong Waves (W), and Presence of Fishermen (F). The Bayes' net graph that represents these relationships is shown below:

R	**P(R)**
+r	0.4
-r	0.6



E	**R**	**P(E | R)**
+e	+r	0.3
-e	+r	0.7
+e	-r	0.6
-e	-r	0.4

W	**R**	**P(W | R)**
+w	+r	0.9
-w	+r	0.1
+w	-r	0.2
-w	-r	0.8

M	**E**	**W**	**P(M | E,W)**
+m	+e	+w	0.1
-m	+e	+w	0.9
+m	+e	-w	0.45
-m	+e	-w	0.55
+m	-e	+w	0.35
-m	-e	+w	0.65
+m	-e	-w	0.9
-m	-e	-w	0.1

F	**W**	**P(F | W)**
+f	+w	0.15
-f	+w	0.85
+f	-w	0.75
-f	-w	0.25

In the following questions we will follow a two-step process:
– (1) Jasmine and Katie observed the random variables $R = -r$ and $F = +f$. We then determine the distribution for $P(M| -r, +f)$ via sampling.

– (2) Based on the estimate for $P(M| -r, +f)$, after committing to an action, landing in the intended location of an action successfully occurs with probability $P(M = +m| -r, +f)$. The other three possible landing positions occur with probability $\frac{P(M=-m|-r,+f)}{3}$ each. Use this transition distribution to calculate the optimal action(s) to take and the expected utility of those actions.

(a) **(i)** **Rejection Sampling**: You want to estimate $P(M = +m | -r, +f)$ by rejection sampling. Below is a list of samples that were generated using prior sampling. Cross out those that would be rejected by rejection sampling.

$\cancel{+r \quad +e \quad +w \quad -m \quad -f}$	$-r \quad -e \quad +w \quad -m \quad +f$
$\cancel{-r \quad -e \quad +w \quad -m \quad -f}$	$\cancel{+r \quad -e \quad +w \quad +m \quad -f}$
$-r \quad +e \quad -w \quad -m \quad +f$	$-r \quad -e \quad -w \quad +m \quad +f$
$\cancel{+r \quad -e \quad -w \quad +m \quad -f}$	$\cancel{+r \quad -e \quad -w \quad +m \quad +f}$
$-r \quad -e \quad -w \quad -m \quad +f$	$-r \quad +e \quad +w \quad -m \quad +f$
$-r \quad +e \quad -w \quad -m \quad +f$	$-r \quad +e \quad -w \quad -m \quad +f$

All samples without the conditioning $-r, +f$ are rejected.

(ii) What is the approximation for $P(M = +m | -r, +f)$ using the remaining samples?
$\frac{1}{7}$, the fraction of accepted samples with $+m$ instantiated.

(iii) What are the optimal action(s) for Jasmine and Katie based on this estimate of $P(M = +m | -r, +f)$?
South, West. As $p(+m| - r, +f) = \frac{1}{7}$, $p(-m| - r, +f) = \frac{6}{7}$. Jasmine and Katie will succeed in the selected action $\frac{1}{7}$ of the time, or take one of the other 3 actions with equal probability of $\frac{2}{7}$. In this case, $p(+m| - r, +f)$ is so low that deciding to head in the direction of the whirlpool actually decreases the chances of landing in it.

(iv) What is the expected utility for the optimal action(s) based on this estimate of $P(M = +m | -r, +f)$?
$\frac{1}{7} * (-50) + \frac{2}{7} * (-50) + \frac{2}{7} * (25) + \frac{2}{7} * (100) = \frac{100}{7}$, the weighted sum of all four outcomes.

(b) **(i)** **Likelihood Weighting:** Suppose instead that you perform likelihood weighting on the following samples to get the estimate for $P(M = +m | -r, +f)$. You receive 4 samples consistent with the evidence.

Sample	**Weight**
$-r \quad -e \quad +w \quad +m \quad +f$	$P(-r)P(+f| + w) = 0.6 * 0.15 = 0.09$
$-r \quad -e \quad -w \quad +m \quad +f$	$P(-r)P(+f| - w) = 0.6 * 0.75 = 0.45$
$-r \quad -e \quad +w \quad -m \quad +f$	$P(-r)P(+f| + w) = 0.6 * 0.15 = 0.09$
$-r \quad +e \quad -w \quad -m \quad +f$	$P(-r)P(+f| - w) = 0.6 * 0.75 = 0.45$

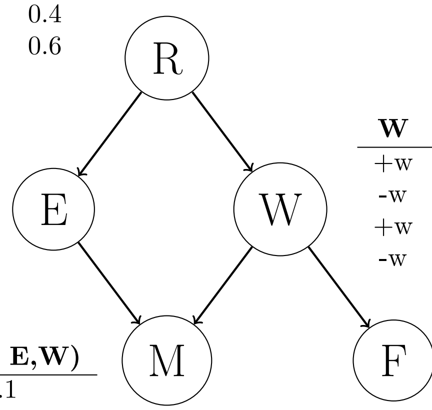
(ii) What is the approximation for $P(M = +m | -r, +f)$ using the samples above?
$\frac{0.09+0.45}{0.09+0.45+0.09+0.45} = \frac{1}{2}$

(iii) What are the optimal action(s) for Jasmine and Katie based on this estimate of $P(M = +m | -r, +f)$?
East

(iv) What is the expected utility for the optimal action(s) based on this estimate of $P(M = +m | -r, +f)$?
$\frac{1}{6} * (-50) + \frac{1}{6} * (-50) + \frac{1}{6} * (25) + \frac{1}{2} * (100) = \frac{75}{2}$

Here is a copy of the Bayes' Net, repeated for your convenience.

R	**P(R)**
+r	0.4
-r	0.6



E	**R**	**P(E | R)**
+e	+r	0.3
-e	+r	0.7
+e	-r	0.6
-e	-r	0.4

W	**R**	**P(W | R)**
+w	+r	0.9
-w	+r	0.1
+w	-r	0.2
-w	-r	0.8

M	**E**	**W**	**P(M | E,W)**
+m	+e	+w	0.1
-m	+e	+w	0.9
+m	+e	-w	0.45
-m	+e	-w	0.55
+m	-e	+w	0.35
-m	-e	+w	0.65
+m	-e	-w	0.9
-m	-e	-w	0.1

F	**W**	**P(F | W)**
+f	+w	0.15
-f	+w	0.85
+f	-w	0.75
-f	-w	0.25

(c) **(i)** **Gibbs Sampling**. Now, we tackle the same problem, this time using Gibbs sampling. We start out with initializing our evidence: $R = -r$, $F = +f$. Furthermore, we start with this random sample:

$-r \;\; +e \;\; -w \;\; +m \;\; +f.$

We select variable E to resample. Calculate the numerical value for:
$P(E = +e|R = -r, W = -w, M = +m, F = +f)$.

$P(E = +e|R = -r, W = -w, M = +m, F = +f) = \frac{P(+e|-r)P(+m|+e,-w)}{P(+e|-r)P(+m|+e,-w)+P(-e|-r)P(+m|-e,-w)}$

$= \frac{0.6*0.45}{0.6*0.45+0.4*0.9} = \frac{3}{7}$

We resample for a long time until we end up with the sample:

$-r \;\; -e \;\; +w \;\; +m \;\; +f.$

Jasmine and Katie are happy for fixing this one sample, but they do not have enough time left to compute another sample before making a move. They will let this one sample approximate the distribution: $P(M = +m| - r, +f)$.

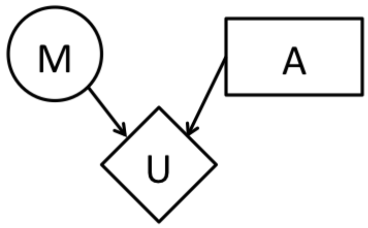
(ii) What is the approximation for $P(M = +m| - r, +f)$, using this one sample?
1

(iii) What are the optimal action(s) for Jasmine and Katie based on this estimate of $P(M = +m| - r, +f)$?
East

(iv) What is the expected utility for the optimal action(s) based on this estimate of $P(M = +m| - r, +f)$?
100

4 . Probability and Decision Networks

The new Josh Bond Movie (M), Skyrise, is premiering later this week. Skyrise will either be great ($+m$) or horrendous ($-m$); there are no other possible outcomes for its quality. Since you are going to watch the movie no matter what, your primary choice is between going to the theater (*theater*) or renting (*rent*) the movie later. Your utility of enjoyment is only affected by these two variables as shown below:



M	P(M)
+m	0.5
-m	0.5

M	A	U(M,A)
+m	*theater*	100
-m	*theater*	10
+m	*rent*	80
-m	*rent*	40

(a) Maximum Expected Utility

Compute the following quantities:

$EU(theater) = P(+m)U(+m, theater) + P(-m)U(-m, theater) = 0.5 * 100 + 0.5 * 10 = 55$

$EU(rent) = P(+m)U(+m, rent) + P(-m)U(-m, rent) = 0.5 * 80 + 0.5 * 40 = 60$

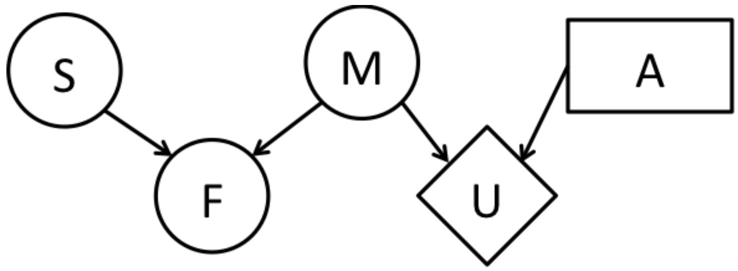
$MEU(\{\}) = 60$

Which action achieves $MEU(\{\}) =$ *rent*

(b) Fish and Chips

Skyrise is being released two weeks earlier in the U.K. than the U.S., which gives you the perfect opportunity to predict the movie's quality. Unfortunately, you don't have access to many sources of information in the U.K., so a little creativity is in order.

You realize that a reasonable assumption to make is that if the movie (M) is great, citizens in the U.K. will celebrate by eating fish and chips (F). Unfortunately the consumption of fish and chips is also affected by a possible food shortage (S), as denoted in the below diagram.



The consumption of fish and chips (F) and the food shortage (S) are both binary variables. The relevant conditional probability tables are listed below:

S	M	F	$P(F\|S,M)$
+s	+m	+f	0.6
+s	+m	-f	0.4
+s	-m	+f	0.0
+s	-m	-f	1.0

S	M	F	$P(F\|S,M)$
-s	+m	+f	1.0
-s	+m	-f	0.0
-s	-m	+f	0.3
-s	-m	-f	0.7

S	$P(S)$
+s	0.2
-s	0.8

You are interested in the value of revealing the food shortage node (S). Answer the following queries:

$EU(theater|+s) =$

The shortage variable is independent of the parents of the utility node when no additional evidence is present; thus, the same values hold:

$EU(theater|+s) = EU(theater) = 55$

$EU(rent|+s) = EU(rent) = 60$

$MEU(\{+s\}) = 60$

Optimal Action Under $\{+s\} = r$ $\quad$ $(Rent)$

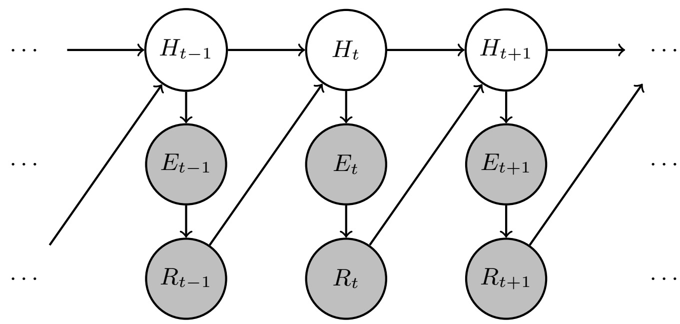
$MEU(\{-s\}) = 60$

Optimal Action Under $\{-s\} = r$ $\quad$ $(Rent)$

$VPI(S) = 0$, since the Value of Perfect Information is the expected difference in MEU given the evidence vs. without the evidence and here the evidence is uninformative.

5 . HMM: Human-Robot Interaction

In the near future, autonomous robots would live among us. Therefore, it is important for the robots to know how to properly act in the presence of humans. In this question, we are exploring a simplified model of this interaction. Here, we are assuming that we can observe the robot's actions at time t, R_t, and an evidence observation, E_t, directly caused by the human action, H_t. Humans actions and Robots actions from the past time-step affect the Human's and Robot's actions in the next time-step. In this problem, we will remain consistent with the convention that capital letters (H_t) refer to random variables and lowercase letters (h_t) refer to a particular value the random variable can take. The structure is given below:



You are supplied with the following probability tables: $P(R_t \mid E_t)$, $P(H_t \mid H_{t-1}, R_{t-1})$, $P(H_0)$, $P(E_t \mid H_t)$.

Let us derive the forward algorithm for this model. We will split our computation into two components, a **time-elapse update** expression and a **observe update** expression.

(a) We would like to incorporate the evidence that we observe at time t. Using the time-lapse update expression we will derive separately, we would like to find the **observe update** expression:

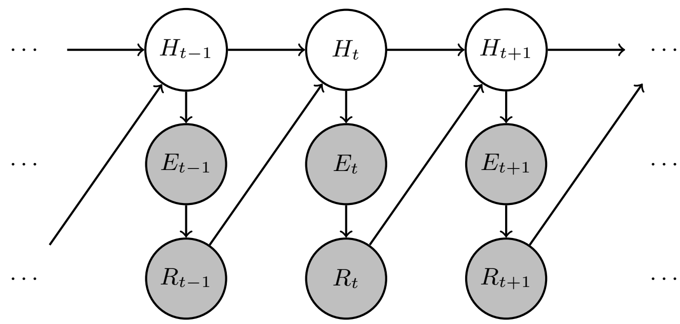
$$O(H_t) = P(H_t | e_{0:t}, r_{0:t})$$

In other words, we would like to compute the distribution of potential human states at time t given all observations up to and including time t. In addition to the conditional probability tables associated with the network's nodes, we are given $T(H_t) = P(H_t \mid e_{0:t-1}, r_{0:t-1})$, which we will assume is correctly computed in the time-elapse update that we will derive in the next part. From the options below, select *all* the options that **both** make valid independence assumptions and would evaluate to the observe update expression.

- [x] $\frac{P(H_t|e_{0:t-1},r_{0:t-1})P(e_t|H_t)P(r_t|e_t)}{\sum_{h_t} P(h_t|e_{0:t-1},r_{0:t-1})P(e_t|h_t)P(r_t|e_t)}$
- [x] $\frac{P(H_t|e_{0:t-1},r_{0:t-1})P(e_t|H_t)}{\sum_{h_t} P(h_t|e_{0:t-1},r_{0:t-1})P(e_t|h_t)}$
- [ ] $\frac{\sum_{e_t} P(H_t|e_{0:t-1},r_{0:t-1})P(e_t|H_t)}{\sum_{h_t} P(h_t|e_{0:t-1},r_{0:t-1})P(e_t|r_{t-1},H_{t-1})}$
- [ ] $\sum_{r_{t-1}} P(H_t \mid e_{0:t-1}, r_{0:t-1})P(r_{t-1} \mid e_{t-1})$
- [ ] $\sum_{r_t} P(H_t \mid e_{0:t-1}, r_{0:t-1})P(r_t \mid r_{t-1}, e_t)$
- [ ] $\sum_{h_{t+1}} P(H_t \mid e_{0:t-1}, r_{0:t-1})P(h_{t+1} \mid r_t)$

$$P(H_t|e_{0:t}, r_{0:t}) = \frac{P(H_t, e_{0:t}, e_{0:t})}{\sum_{h_t} P(h_t, e_{0:t}, e_{0:t})} = \frac{P(H_t \mid e_{0:t-1}, r_{0:t-1})P(e_t \mid H_t)P(r_t \mid e_t)}{P(r_t \mid e_t)\sum_{h_t} P(h_t \mid e_{0:t-1}, r_{0:t-1})P(e_t \mid h_t)}$$

The structure below is identical to the one in the beginning of the question and is repeated for your convenience.



(b) We are interested in predicting what the state of human is at time t (H_t), given all the observations through $t-1$. Therefore, the **time-elapse update** expression has the following form:

$$T(H_t) = P(H_t | e_{0:t-1}, r_{0:t-1})$$

Derive an expression for the given time-elapse update above using the probability tables provided in the question and the observe update expression, $O(H_{t-1}) = P(H_{t-1}|e_{0:t-1}, r_{0:t-1})$. Write your final expression in the space provided at below. You may use the function O in your solution if you prefer.

The derivation of the time-elapse update for this setup is similar to the one we have seen in lecture; however, here, we have additional observations and dependencies.

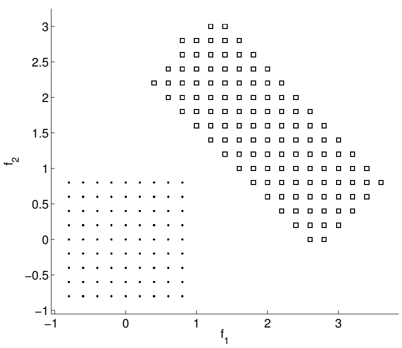
$$\begin{aligned} P(H_t|e_{0:t-1}, r_{0:t-1}) &= \sum_{h_{t-1}} P(H_t, h_{t-1} \mid e_{0:t-1}, r_{0:t-1}) \\ &= \sum_{h_{t-1}} P(H_t \mid h_{t-1}, r_{t-1}) P(h_{t-1} \mid e_{0:t-1}, r_{0:t-1}) \end{aligned}$$

$P(H_t|e_{0:t-1}, r_{0:t-1}) =$ $\sum_{h_{t-1}} P(H_t \mid h_{t-1}, r_{t-1}) P(h_{t-1} \mid e_{0:t-1}, r_{0:t-1})$

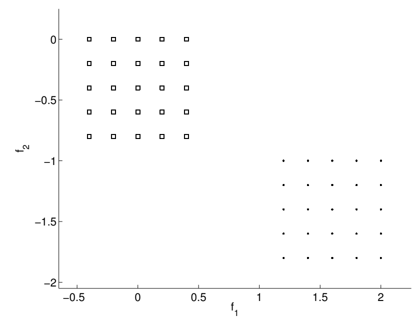
6 . Naïve Bayes Modeling Assumptions

You are given points from 2 classes, shown as rectangles and dots. For each of the following sets of points, mark if they satisfy all the Naïve Bayes modelling assumptions, or they do not satisfy all the Naïve Bayes modelling assumptions. Note that in (c), 4 rectangles overlap with 4 dots.

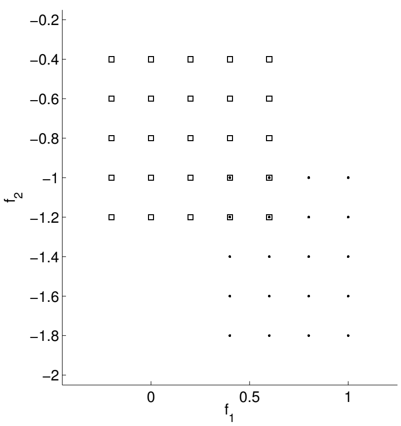
The conditional independence assumptions made by the Naïve Bayes model are that features are conditionally independent when given the class. Features being independent once the class label is known means that for a fixed class the distribution for f_1 cannot depend on f_2, and the other way around. Concretely, for discrete-valued features as shown below, this means each class needs to have a distribution that corresponds to an axis-aligned rectangle. No other assumption is made by the Naïve Bayes model. Note that linear separability is not an assumption of the Naïve Bayes model—what is true is that for a Naïve Bayes model with all binary variables the decision boundary between the two classes is a hyperplane (i.e., it's a linear classifier). That, however, wasn't relevant to the question as the question examined which probability distribution a Naïve Bayes model can represent, not which decision boundaries.



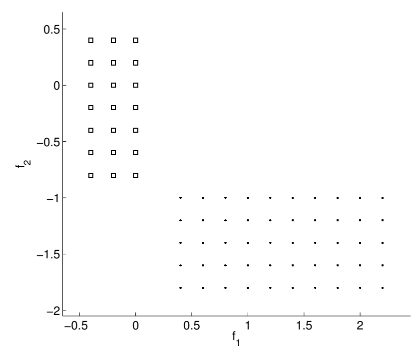
(a) ○ Satisfies ● Does not Satisfy



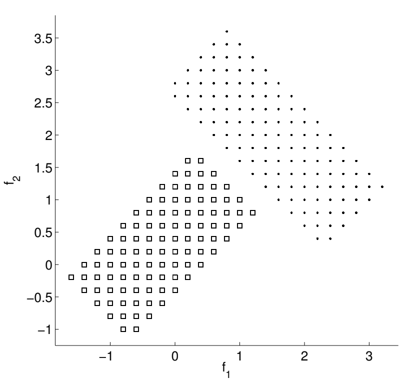
(b) ● Satisfies ○ Does not Satisfy



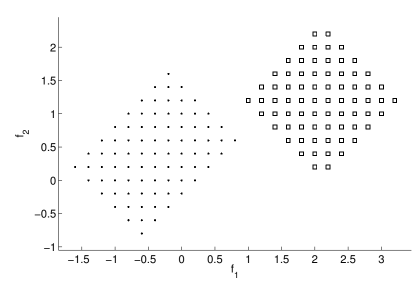
(c) ● Satisfies ○ Does not Satisfy



(d) ● Satisfies ○ Does not Satisfy



(e) ○ Satisfies ● Does not Satisfy



(f) ○ Satisfies ● Does not Satisfy

A note about feature independence: The Naïve Bayes model assumes features are conditionally independent given the class. Why does this result in axis-aligned rectangles for discrete feature distributions? Intuitively, this is because fixing one value is uninformative about the other: within a class, the values of one feature are constant across the other. For instance, the dark square class in (b) has $f_1 \in [-0.5, 0.5]$ and $f_2 \in [-1, 0]$ and fixing one has no impact on the domain of the other. However, when the features of a class are not axis-aligned then fixing one limits the domain of the other, inducing dependence. In (e), fixing $f_2 = 1.5$ restricts f_1 to the two points at the top, whereas fixing $f_2 = 0$ gives a larger domain.