Q1. Bayes Nets and Joint Distributions

(a) Write down the joint probability distribution associated with the following Bayes Net. Express the answer as a product of terms representing individual conditional probabilities tables associated with this Bayes Net:

```
A -> B
|  |  \\
|  | /
C | |
|  |  \\
|  | /
|  |  \\
|  | /
D
```

(b) Draw the Bayes net associated with the following joint distribution:

\[ P(A) \cdot P(B) \cdot P(C|A, B) \cdot P(D|C) \cdot P(E|B, C) \]

(c) Do the following products of factors correspond to a valid joint distribution over the variables A, B, C, D? (Circle TRUE/FALSE.)

(i) TRUE FALSE \[ P(A) \cdot P(B) \cdot P(C|A) \cdot P(C|B) \cdot P(D|C) \]

(ii) TRUE FALSE \[ P(A) \cdot P(B|A) \cdot P(C) \cdot P(D|B, C) \]

(iii) TRUE FALSE \[ P(A) \cdot P(B|A) \cdot P(C) \cdot P(C|A) \cdot P(D) \]

(iv) TRUE FALSE \[ P(A|B) \cdot P(B|C) \cdot P(C|D) \cdot P(D|A) \]
(d) What factor can be multiplied with the following factors to form a valid joint distribution? (Write “none” if the given set of factors can’t be turned into a joint by the inclusion of exactly one more factor.)

(i) \( P(A) \cdot P(B|A) \cdot P(C|A) \cdot P(E|B, C, D) \)

(ii) \( P(D) \cdot P(B) \cdot P(C|D, B) \cdot P(E|C, D, A) \)

(e) Answer the next questions based off of the Bayes Net below:

All variables have domains of \(-1, 0, 1\)

(i) Before eliminating any variables or including any evidence, how many entries does the factor at \(G\) have?

(ii) Now we observe \(e = 1\) and want to query \(P(D|e = 1)\), and you get to pick the first variable to be eliminated.

- Which choice would create the largest factor \(f_1\)?

- Which choice would create the smallest factor \(f_1\)?
Q2. Bayes’ Nets Representation

(a) Graph Structure: Conditional Independence

Consider the Bayes’ net given below.

```
A  B  C
F  H  A
  A  A
  G  D
  E
F  G  H
```

Remember that \( X \perp \perp Y \) reads as “\( X \) is independent of \( Y \) given nothing”, and \( X \perp \perp Y|\{Z,W\} \) reads as “\( X \) is independent of \( Y \) given \( Z \) and \( W \).”

For each expression, fill in the corresponding circle to indicate whether it is True or False.

(i) True False It is guaranteed that \( A \perp \perp B \)

(ii) True False It is guaranteed that \( A \perp \perp C \)

(iii) True False It is guaranteed that \( A \perp \perp D | \{B,H\} \)

(iv) True False It is guaranteed that \( A \perp \perp E|F \)

(v) True False It is guaranteed that \( G \perp \perp E|B \)

(vi) True False It is guaranteed that \( F \perp \perp C|D \)

(vii) True False It is guaranteed that \( E \perp \perp D|B \)

(viii) True False It is guaranteed that \( C \perp \perp H|G \)
(b) Graph structure: Representational Power

Recall that any directed acyclic graph $G$ has an associated family of probability distributions, which consists of all probability distributions that can be represented by a Bayes’ net with structure $G$.

For the following questions, consider the following six directed acyclic graphs:

(i) Assume all we know about the joint distribution $P(A, B, C)$ is that it can be represented by the product $P(A|B, C)P(B|C)P(C)$. Mark each graph for which the associated family of probability distributions is guaranteed to include $P(A, B, C)$.

□ $G_1$ □ $G_2$ □ $G_3$
□ $G_4$ □ $G_5$ □ $G_6$

(ii) Now assume all we know about the joint distribution $P(A, B, C)$ is that it can be represented by the product $P(C|B)P(B|A)P(A)$. Mark each graph for which the associated family of probability distributions is guaranteed to include $P(A, B, C)$.

□ $G_1$ □ $G_2$ □ $G_3$  
□ $G_4$ □ $G_5$ □ $G_6$
(c) Marginalization and Conditioning

Consider a Bayes’ net over the random variables $A, B, C, D, E$ with the structure shown below, with full joint distribution $P(A, B, C, D, E)$.

The following three questions describe different, unrelated situations (your answers to one question should not influence your answer to other questions).

(i) Consider the marginal distribution $P(A, B, D, E) = \sum_c P(A, B, c, D, E)$, where $C$ was eliminated. On the diagram below, draw the minimal number of arrows that results in a Bayes’ net structure that is able to represent this marginal distribution. If no arrows are needed write “No arrows needed.”

(ii) Assume we are given an observation: $A = a$. On the diagram below, draw the minimal number of arrows that results in a Bayes’ net structure that is able to represent the conditional distribution $P(B, C, D, E \mid A = a)$. If no arrows are needed write “No arrows needed.”

(iii) Assume we are given two observations: $D = d, E = e$. On the diagram below, draw the minimal number of arrows that results in a Bayes’ net structure that is able to represent the conditional distribution $P(A, B, C \mid D = d, E = e)$. If no arrows are needed write “No arrows needed.”