Regular Discussion 12

1 Optimization

We would like to classify some data. We have N samples, where each sample consists of a feature vector $\mathbf{x} = [x_1, \cdots, x_k]^T$ and a label $y \in \{0, 1\}$.

Logistic regression produces predictions as follows:

$$P(Y = 1 \mid X) = h(\mathbf{x}) = s\left(\sum_i w_i x_i\right) = \frac{1}{1 + \exp(-(\sum_i w_i x_i))}$$

$$s(\gamma) = \frac{1}{1 + \exp(-\gamma)}$$

where $s(\gamma)$ is the logistic function, $\exp x = e^x$, and $\mathbf{w} = [w_1, \cdots, w_k]^T$ are the learned weights.

Let's find the weights w_j for logistic regression using stochastic gradient descent. We would like to minimize the following loss function (called the cross-entropy loss) for each sample:

$$L = -[y \ln h(\mathbf{x}) + (1 - y) \ln(1 - h(\mathbf{x}))]$$

(a) Show that $s'(\gamma) = s(\gamma)(1 - s(\gamma))$

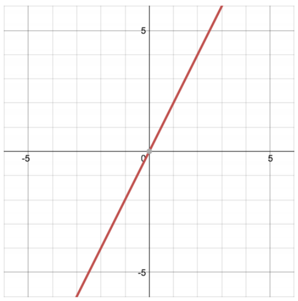
(b) Find $\frac{dL}{dw_j}$. Use the fact from the previous part.

(c) Now, find a simple expression for $\nabla_{\mathbf{w}} L = [\frac{dL}{dw_1}, \frac{dL}{dw_2}, ..., \frac{dL}{dw_k}]^T$

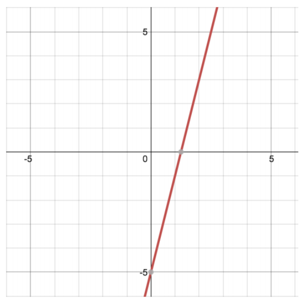
(d) Write the stochastic gradient descent update for $\mathbf{w}$. Our step size is η.

2 Neural Network Representations

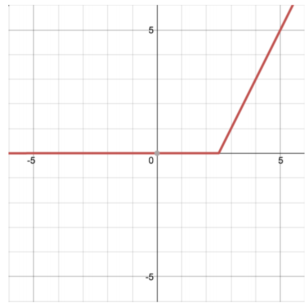
You are given a number of functions (a-h) of a single variable, x, which are graphed below. The computation graphs on the following pages will start off simple and get more complex, building up to neural networks. For each computation graph, indicate which of the functions below they are able to represent.



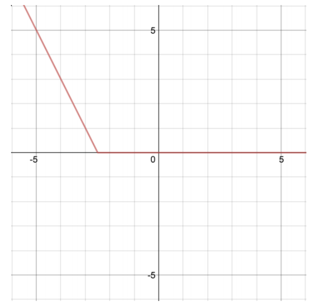
(a) $2x$



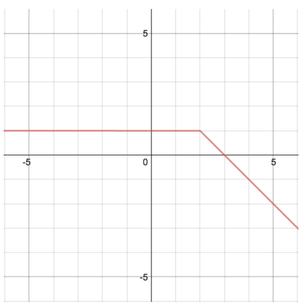
(b) $4x - 5$



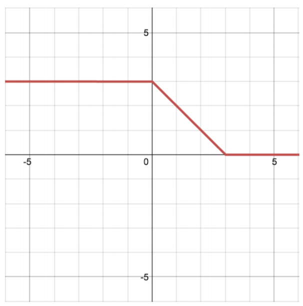
(c) $\begin{cases} 2x - 5 & x \geq 2.5 \\ 0 & x < 2.5 \end{cases}$



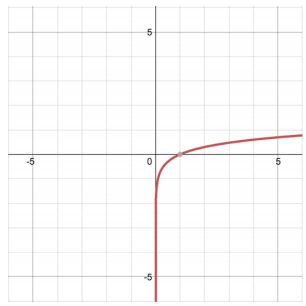
(d) $\begin{cases} -2x - 5 & x \leq -2.5 \\ 0 & x > -2.5 \end{cases}$



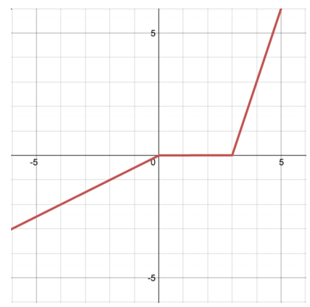
(e) $\begin{cases} -x + 3 & x \geq 2 \\ 1 & x < 2 \end{cases}$



(f) $\begin{cases} 3 & x \leq 0 \\ 3 - x & 0 < x \leq 3 \\ 0 & x > 3 \end{cases}$

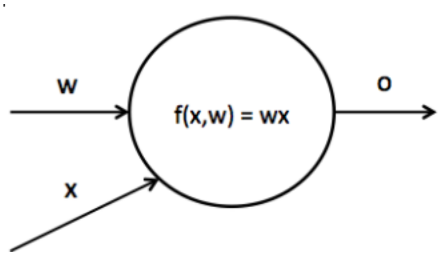


(g) $\log(x)$

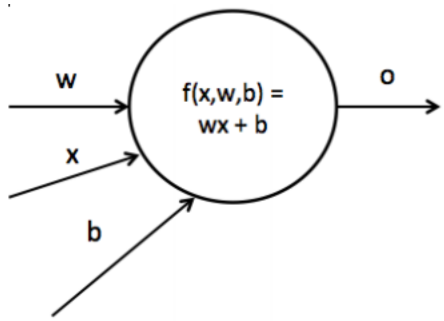


(h) $\begin{cases} 0.5x & x \leq 0 \\ 0 & 0 < x \leq 3 \\ 3x - 9 & x > 3 \end{cases}$

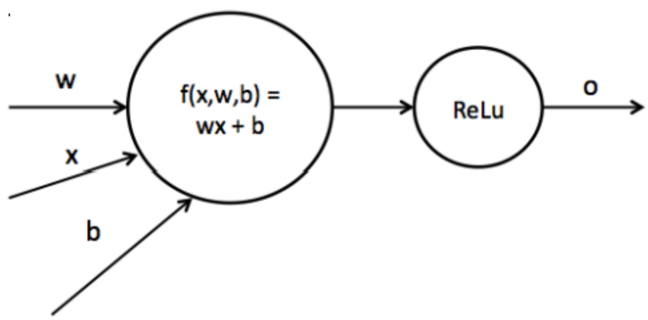
1. Consider the following computation graph, computing a linear transformation with scalar input x, weight w, and output o, such that $o = wx$. Which of the funcions can be represented by this graph? For the options which can, write out the appropriate value of w.



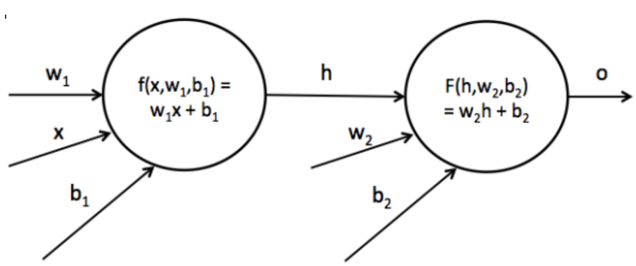
2. Now we introduce a bias term b into the graph, such that $o = wx + b$ (this is known as an *affine* function). Which of the functions can be represented by this network? For the options which can, write out an appropriate value of w, b.



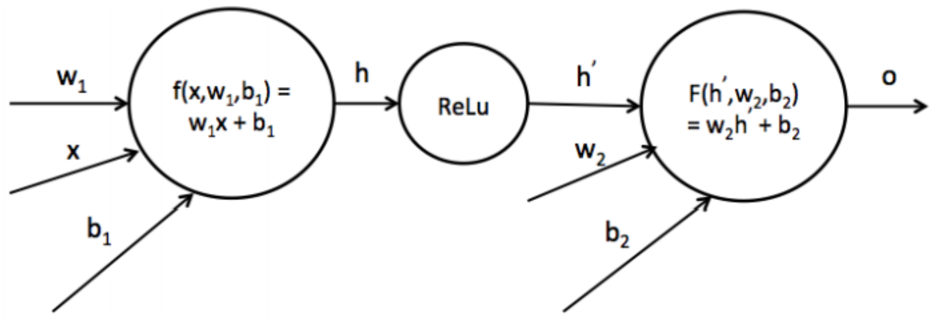
3. We can introduce a non-linearity into the network as indicated below. We use the ReLU non-linearity, which has the form $ReLU(x) = \max(0, x)$. Now which of the functions can be represented by this neural network with weight w and bias b? For the options which can, write out an appropriate value of w, b.



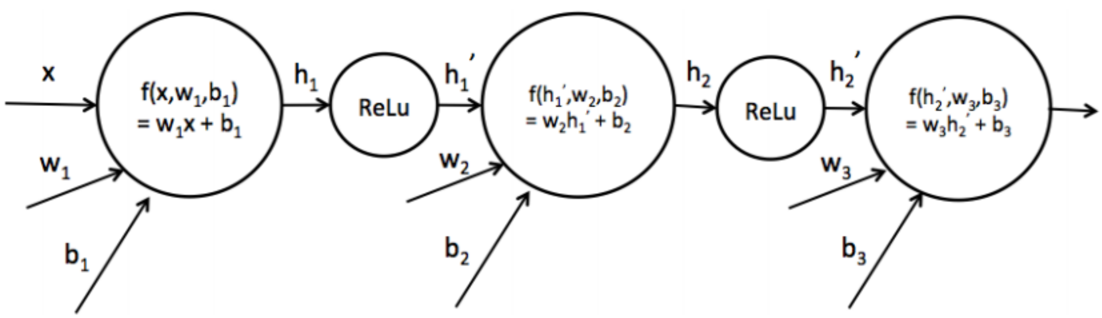
4. Now we consider neural networks with multiple affine transformations, as indicated below. We now have two sets of weights and biases w_1, b_1 and w_2, b_2. We denote the result of the first transformation h such that $h = w_1 x + b_1$, and $o = w_2 h + b_2$. Which of the functions can be represented by this network? For the options which can, write out appropriate values of w_1, w_2, b_1, b_2.



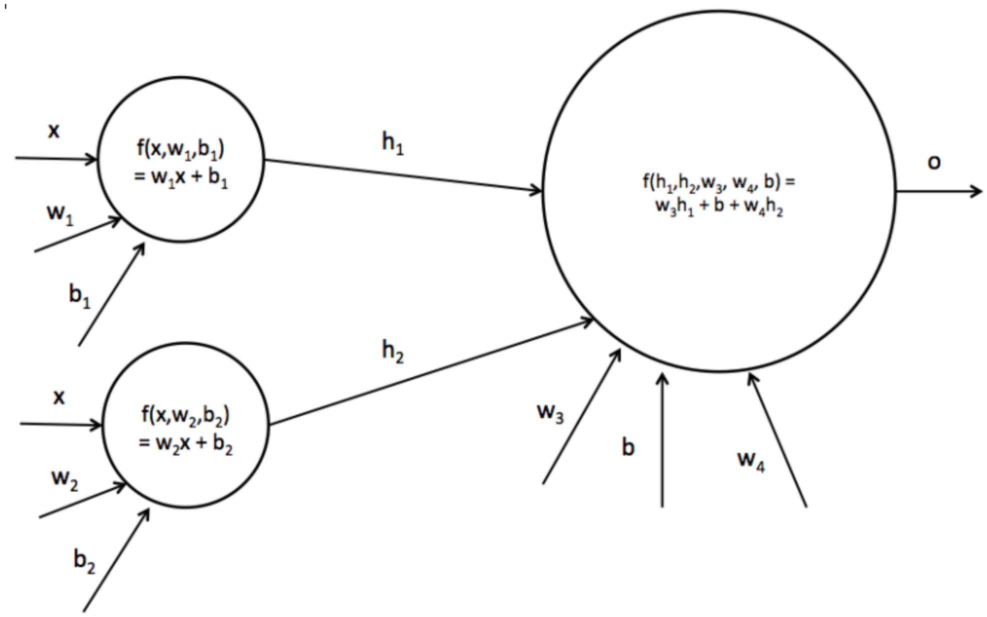
5. Next we add a ReLU non-linearity to the network after the first affine transformation, creating a hidden layer. Which of the functions can be represented by this network? For the options which can, write out appropriate values of w_1, w_2, b_1, b_2.



6. Now we add another hidden layer to the network, as indicated below. Which of the functions can be represented by this network?



7. We'd like to consider using a neural net with just one hidden layer, but have it be larger – a hidden layer of size 2. Let's first consider using just two affine functions, with no nonlinearity in between. Which of the functions can be represented by this network?



8. Now we'll add a non-linearity between the two affine layers, to produce the neural network below with a hidden layer of size 2. Which of the functions can be represented by this network?