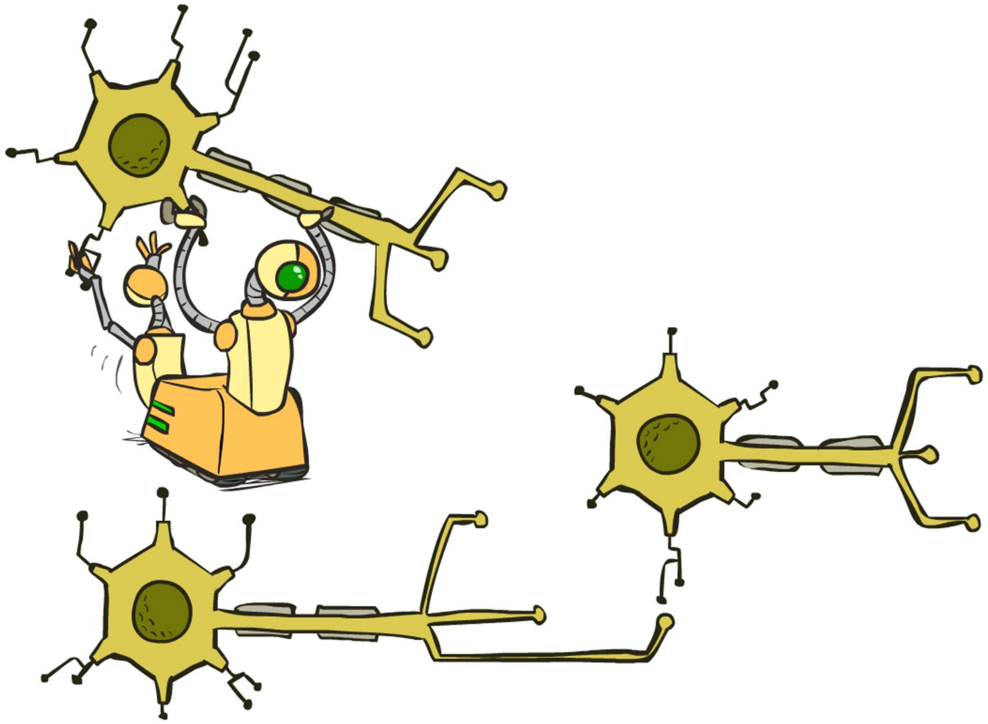
CS 188: Artificial Intelligence

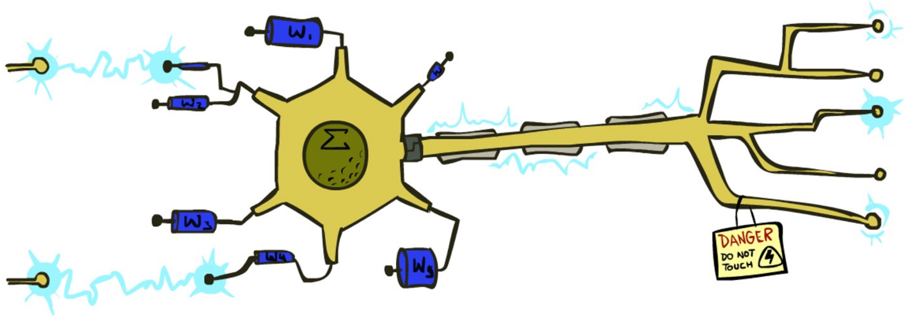
Optimization and Neural Networks



[These slides were created by Dan Klein, Pieter Abbeel, Anca Dragan for CS188 Intro to AI at UC Berkeley. All CS188 materials are available at http://ai.berkeley.edu.]

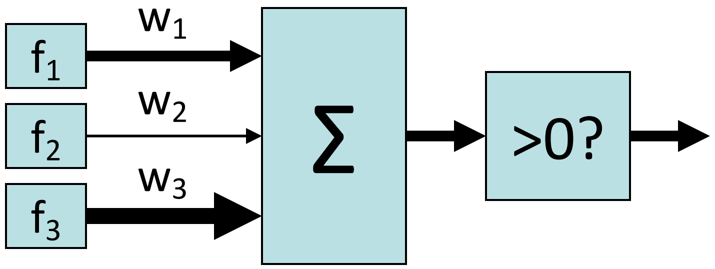
Reminder: Linear Classifiers

- Inputs are feature values
- Each feature has a weight
- Sum is the activation



$$\text{activation}_w(x) = \sum_i w_i \cdot f_i(x) = w \cdot f(x)$$

- If the activation is:
  - Positive, output +1
  - Negative, output -1



How to get probabilistic decisions?

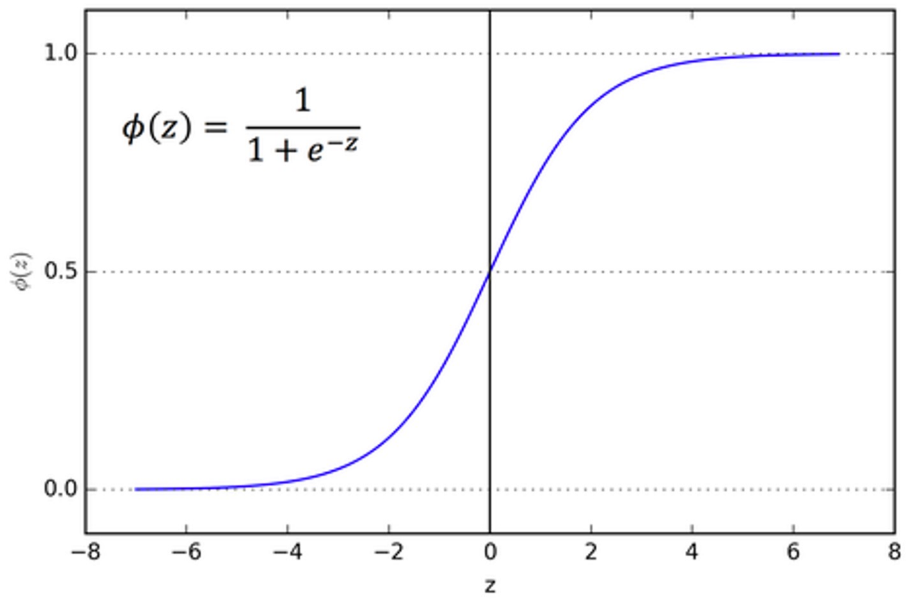
Activation: $z = w \cdot f(x)$

If $z = w \cdot f(x)$ very positive → want probability going to 1

If $z = w \cdot f(x)$ very negative → want probability going to 0

Sigmoid function

$$\phi(z) = \frac{1}{1 + e^{-z}}$$



Best w?

Maximum likelihood estimation:

$$\max_w \quad ll(w) = \max_w \quad \sum_i \log P(y^{(i)}|x^{(i)}; w)$$

with: $$P(y^{(i)} = +1|x^{(i)}; w) = \frac{1}{1 + e^{-w \cdot f(x^{(i)})}}$$

$$P(y^{(i)} = -1|x^{(i)}; w) = 1 - \frac{1}{1 + e^{-w \cdot f(x^{(i)})}}$$

= Logistic Regression

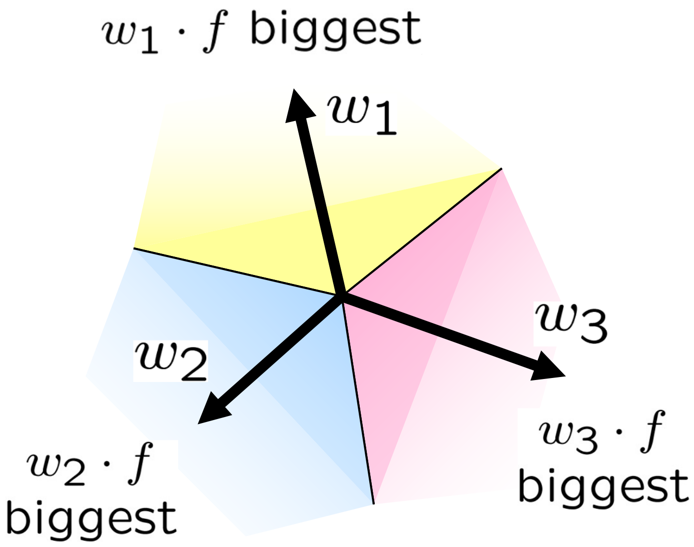
Multiclass Logistic Regression

Multi-class linear classification

A weight vector for each class: w_y

Score (activation) of a class y: $w_y \cdot f(x)$

Prediction w/highest score wins: $y = \arg\max_y \; w_y \cdot f(x)$



How to make the scores into probabilities?

$$z_1, z_2, z_3 \rightarrow \frac{e^{z_1}}{e^{z_1}+e^{z_2}+e^{z_3}}, \frac{e^{z_2}}{e^{z_1}+e^{z_2}+e^{z_3}}, \frac{e^{z_3}}{e^{z_1}+e^{z_2}+e^{z_3}}$$

original activations

softmax activations

Best w?

Maximum likelihood estimation:

$$\max_w \quad ll(w) = \max_w \quad \sum_i \log P(y^{(i)}|x^{(i)}; w)$$

with:

$$P(y^{(i)}|x^{(i)}; w) = \frac{e^{w_{y^{(i)}} \cdot f(x^{(i)})}}{\sum_y e^{w_y \cdot f(x^{(i)})}}$$

= Multi-Class Logistic Regression

This Lecture

Optimization

i.e., how do we solve:

$$\max_{w} \; ll(w) = \max_{w} \; \sum_{i} \log P(y^{(i)} | x^{(i)}; w)$$

Hill Climbing

- Recall from CSPs lecture: simple, general idea
  - Start wherever
  - Repeat: move to the best neighboring state
  - If no neighbors better than current, quit

- What's particularly tricky when hill-climbing for multiclass logistic regression?
  - Optimization over a continuous space
    - Infinitely many neighbors!
    - How to do this efficiently?

Review: Derivatives and Gradients

- What is the derivative of the function $g(x) = x^2 + 3$?

$$\frac{dg}{dx} = 2x$$

- What is the derivative of g(x) at x=5?

$$\frac{dg}{dx}\Big|_{x=5} = 10$$

Review: Derivatives and Gradients

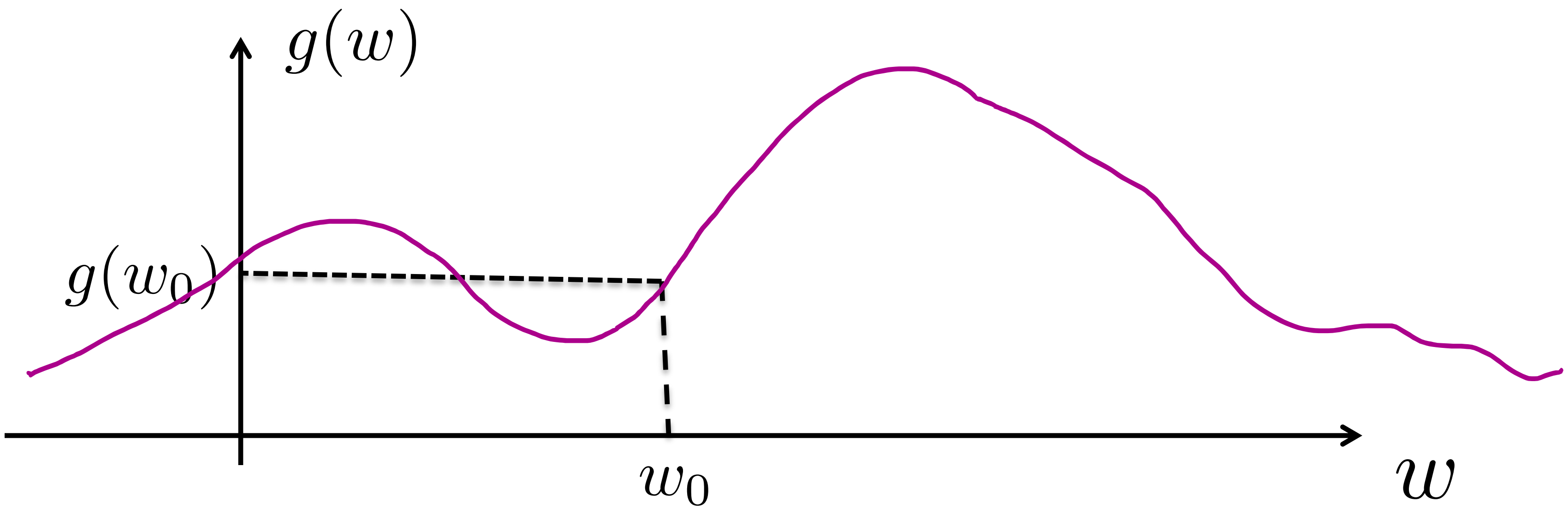
- What is the gradient of the function $g(x, y) = x^2 y$?
  - Recall: Gradient is a vector of partial derivatives with respect to each variable

$$\nabla g = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \\ \frac{\partial f}{\partial y} \end{bmatrix} = \begin{bmatrix} 2xy \\ \\ x^2 \end{bmatrix}$$

- What is the derivative of g(x, y) at x=0.5, y=0.5?

$$\nabla g|_{x=0.5, y=0.5} = \begin{bmatrix} 2(0.5)(0.5) \\ (0.5^2) \end{bmatrix} = \begin{bmatrix} 0.5 \\ 0.25 \end{bmatrix}$$

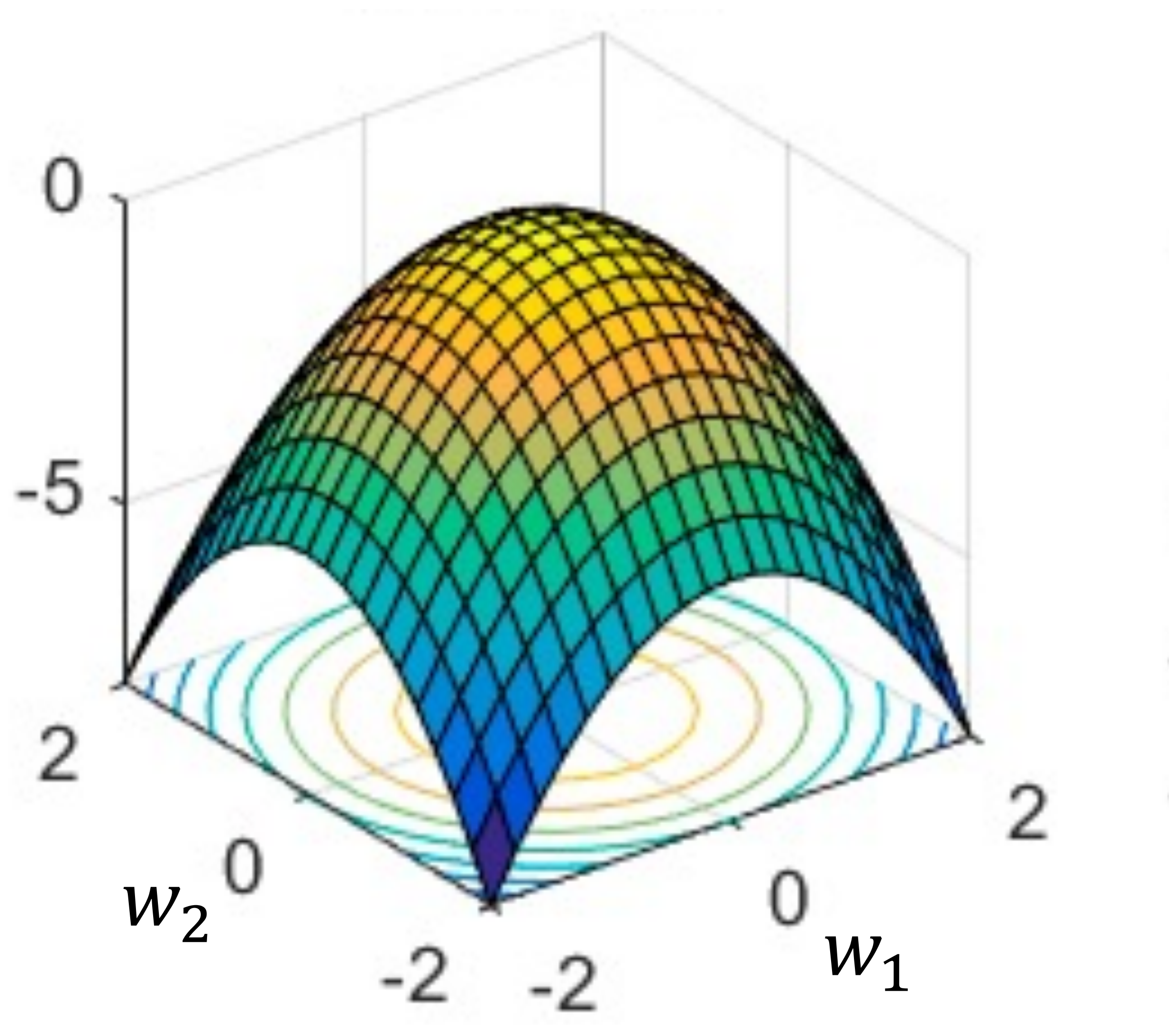
1-D Optimization



- Could evaluate $g(w_0 + h)$ and $g(w_0 - h)$
  - Then step in best direction

- Or, evaluate derivative: $$\frac{\partial g(w_0)}{\partial w} = \lim_{h \to 0} \frac{g(w_0 + h) - g(w_0 - h)}{2h}$$
  - Tells which direction to step into

2-D Optimization



Source: offconvex.org

Gradient Ascent

- Perform update in uphill direction for each coordinate
- The steeper the slope (i.e. the higher the derivative) the bigger the step for that coordinate
- E.g., consider: $g(w_1, w_2)$
  - Updates:

$$w_1 \leftarrow w_1 + \alpha * \frac{\partial g}{\partial w_1}(w_1, w_2)$$

$$w_2 \leftarrow w_2 + \alpha * \frac{\partial g}{\partial w_2}(w_1, w_2)$$

- Updates in vector notation:

$$w \leftarrow w + \alpha * \nabla_w g(w)$$

with: $\nabla_w g(w) = \begin{bmatrix} \frac{\partial g}{\partial w_1}(w) \\ \frac{\partial g}{\partial w_2}(w) \end{bmatrix}$ **= gradient**

Gradient Ascent

- Idea:
  - Start somewhere
  - Repeat: Take a step in the gradient direction

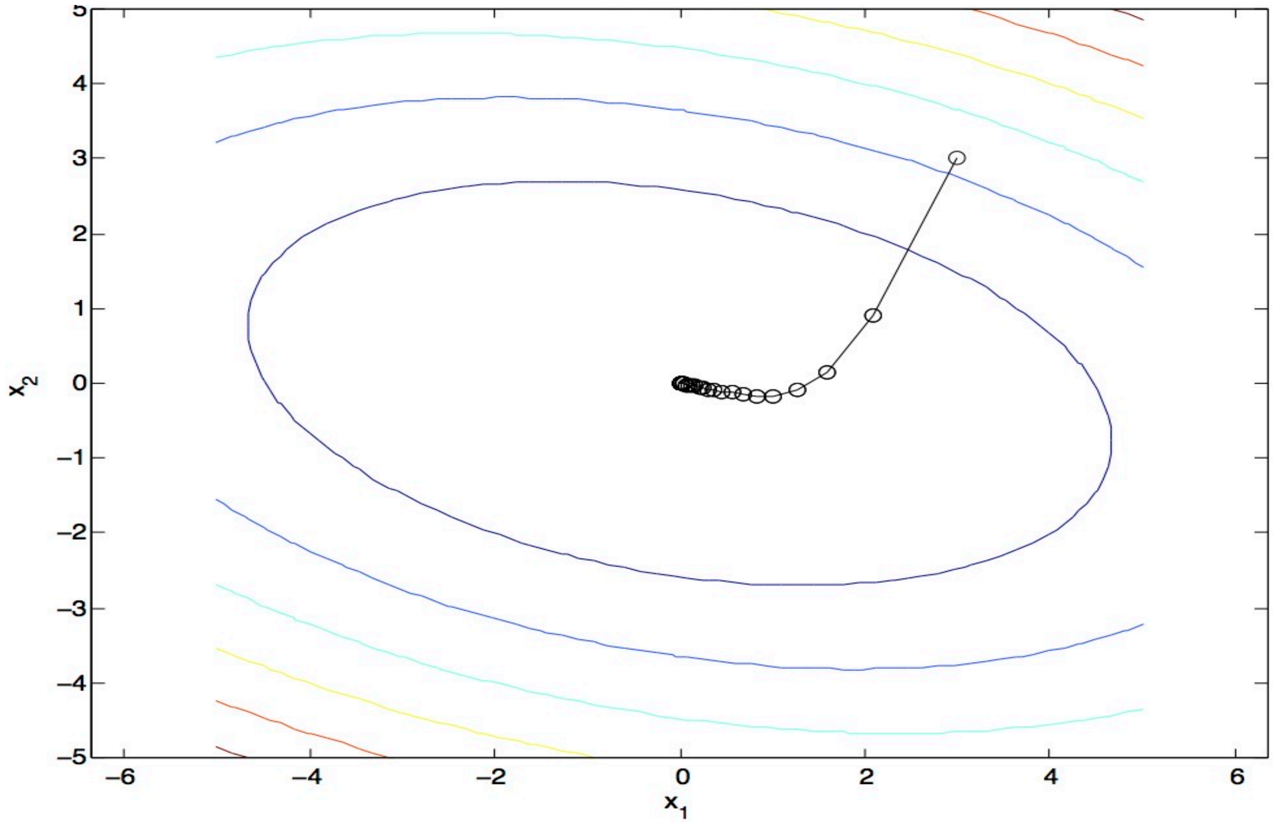


Figure source: Mathworks

Gradient Ascent

- Idea:
  - Start somewhere
  - Repeat: Take a step in the gradient direction

Not guaranteed to find global maximum:

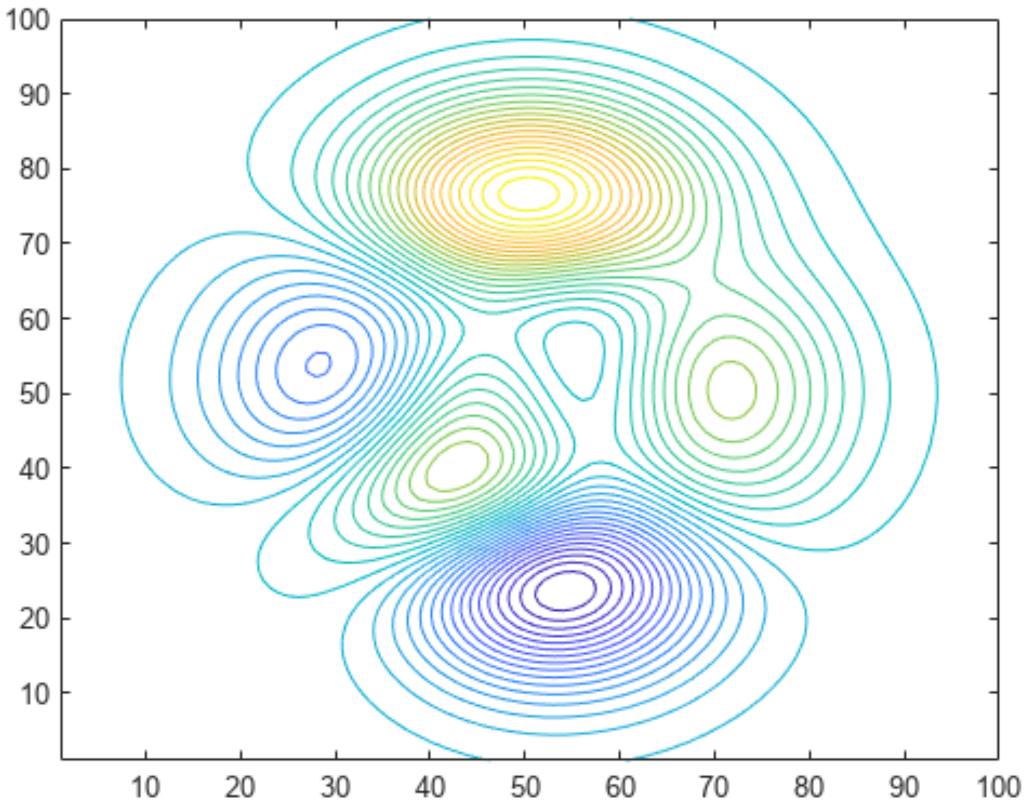
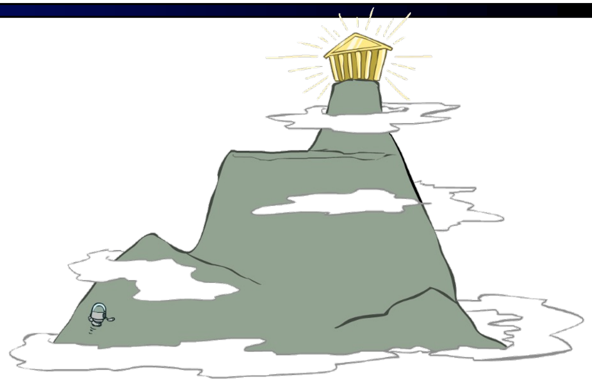


Figure source: Mathworks

What is the Steepest Direction?*

$$\max_{\Delta:\Delta_1^2+\Delta_2^2\leq\varepsilon} \quad g(w+\Delta)$$



- First-Order Taylor Expansion: $$g(w+\Delta)\approx g(w)+\frac{\partial g}{\partial w_1}\Delta_1+\frac{\partial g}{\partial w_2}\Delta_2$$
- Steepest Descent Direction: $$\max_{\Delta:\Delta_1^2+\Delta_2^2\leq\varepsilon} \quad g(w)+\frac{\partial g}{\partial w_1}\Delta_1+\frac{\partial g}{\partial w_2}\Delta_2$$
- Recall: $$\max_{\Delta:\|\Delta\|\leq\varepsilon} \Delta^\top a \quad \rightarrow \quad \Delta=\varepsilon\frac{a}{\|a\|}$$
- Hence, solution: $$\Delta=\varepsilon\frac{\nabla g}{\|\nabla g\|}$$ **Gradient direction = steepest direction!** $$\nabla g=\begin{bmatrix}\frac{\partial g}{\partial w_1}\\ \frac{\partial g}{\partial w_2}\end{bmatrix}$$

Gradient in n dimensions

$$\nabla g = \begin{bmatrix} \frac{\partial g}{\partial w_1} \\ \frac{\partial g}{\partial w_2} \\ \cdots \\ \frac{\partial g}{\partial w_n} \end{bmatrix}$$

Optimization Procedure: Gradient Ascent

```
Init w
for iter = 1, 2, …
```

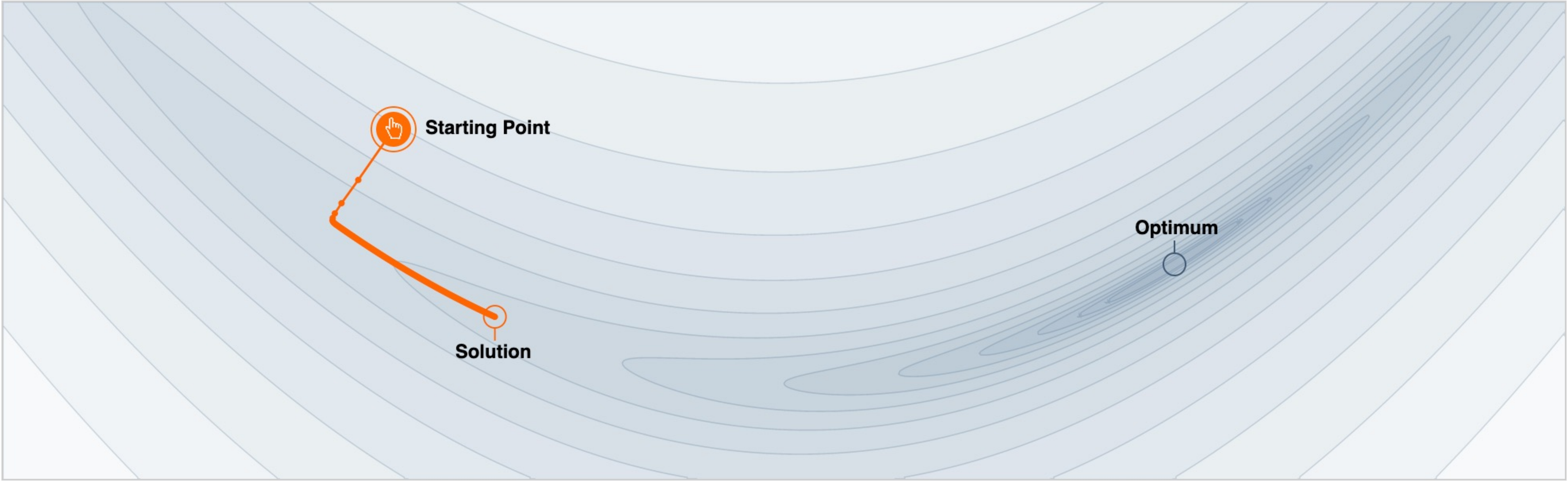
$$w \leftarrow w + \alpha \cdot \nabla g(w)$$

- α: learning rate --- tweaking parameter that needs to be chosen carefully
- How? Try multiple choices
  - Crude rule of thumb: update changes w about 0.1 – 1 %

Learning Rate

Choice of learning rate α is a hyperparameter

Example: $\alpha=0.001$ (too small)

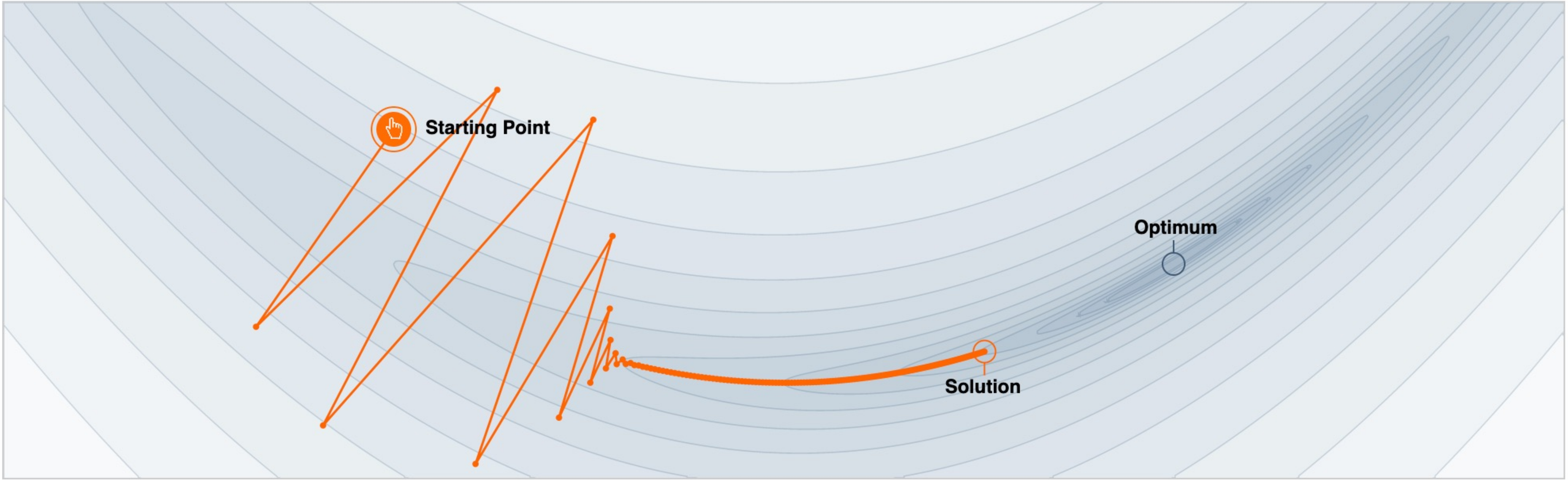


Source: https://distill.pub/2017/momentum/

Learning Rate

Choice of step size α is a hyperparameter

Example: $\alpha=0.004$ (too large)



Source: https://distill.pub/2017/momentum/

Gradient Ascent with Momentum*

- Often use *momentum* to improve gradient ascent convergence

Gradient Ascent:

```
Init w
for iter = 1, 2, …
```

$$w \leftarrow w + \alpha \cdot \nabla g(w)$$

Gradient Ascent with momentum:

```
Init w
for iter = 1, 2, …
```

$$z \leftarrow \beta \cdot z + \nabla g(w)$$

$$w \leftarrow w + \alpha \cdot z$$

- One interpretation: *w* moves like a particle with mass
- Another: *exponential moving average* on gradient

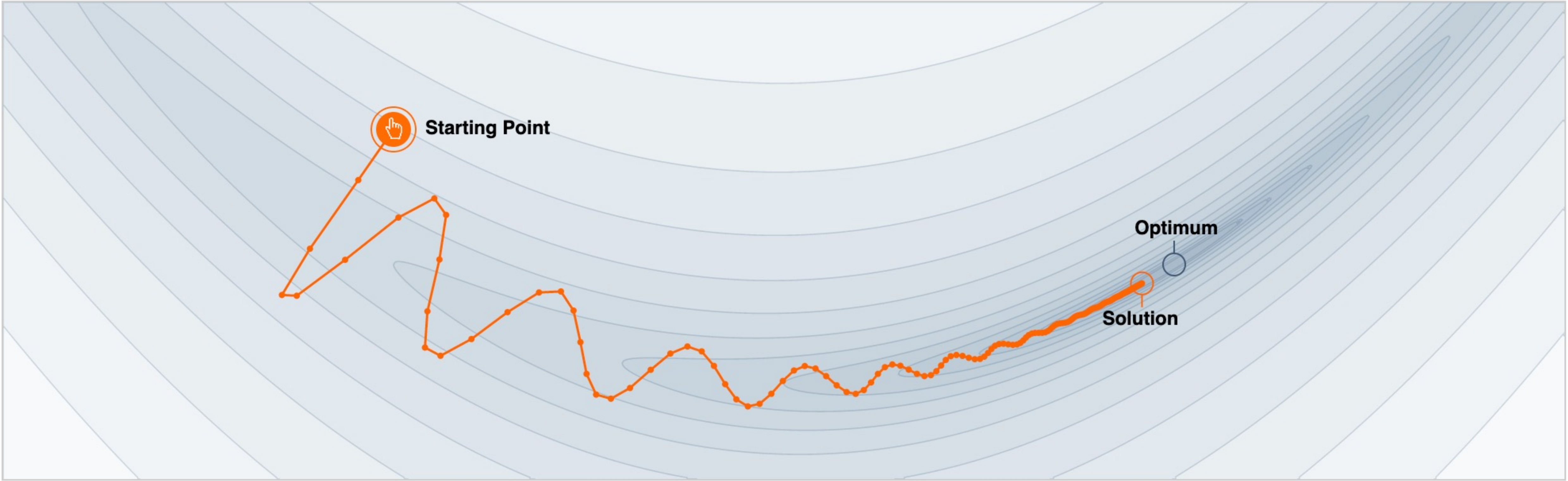
Gradient Ascent with Momentum*

Example: α=0.001 and β=0.0

Source: https://distill.pub/2017/momentum/

Gradient Ascent with Momentum*

Example: α=0.001 and β=0.9



Source: https://distill.pub/2017/momentum/

Batch Gradient Ascent on the Log Likelihood Objective

$$\max_w \; ll(w) = \max_w \; \underbrace{\sum_i \log P(y^{(i)}|x^{(i)};w)}_{g(w)}$$

- init w
- for iter = 1, 2, …

$$w \leftarrow w + \alpha * \sum_i \nabla \log P(y^{(i)}|x^{(i)};w)$$

Stochastic Gradient Ascent on the Log Likelihood Objective

$$\max_w \quad ll(w) = \max_w \quad \sum_i \log P(y^{(i)}|x^{(i)}; w)$$

Observation: once gradient on one training example has been computed, might as well incorporate before computing next one

- `init` w
- `for iter = 1, 2, …`
  - `pick random j`

    $$w \leftarrow w + \alpha * \nabla \log P(y^{(j)}|x^{(j)}; w)$$

Mini-Batch Gradient Ascent on the Log Likelihood Objective

$$\max_w \; ll(w) = \max_w \; \sum_i \log P(y^{(i)}|x^{(i)};w)$$

Observation: gradient over small set of training examples (=mini-batch) can be computed in parallel, might as well do that instead of a single one

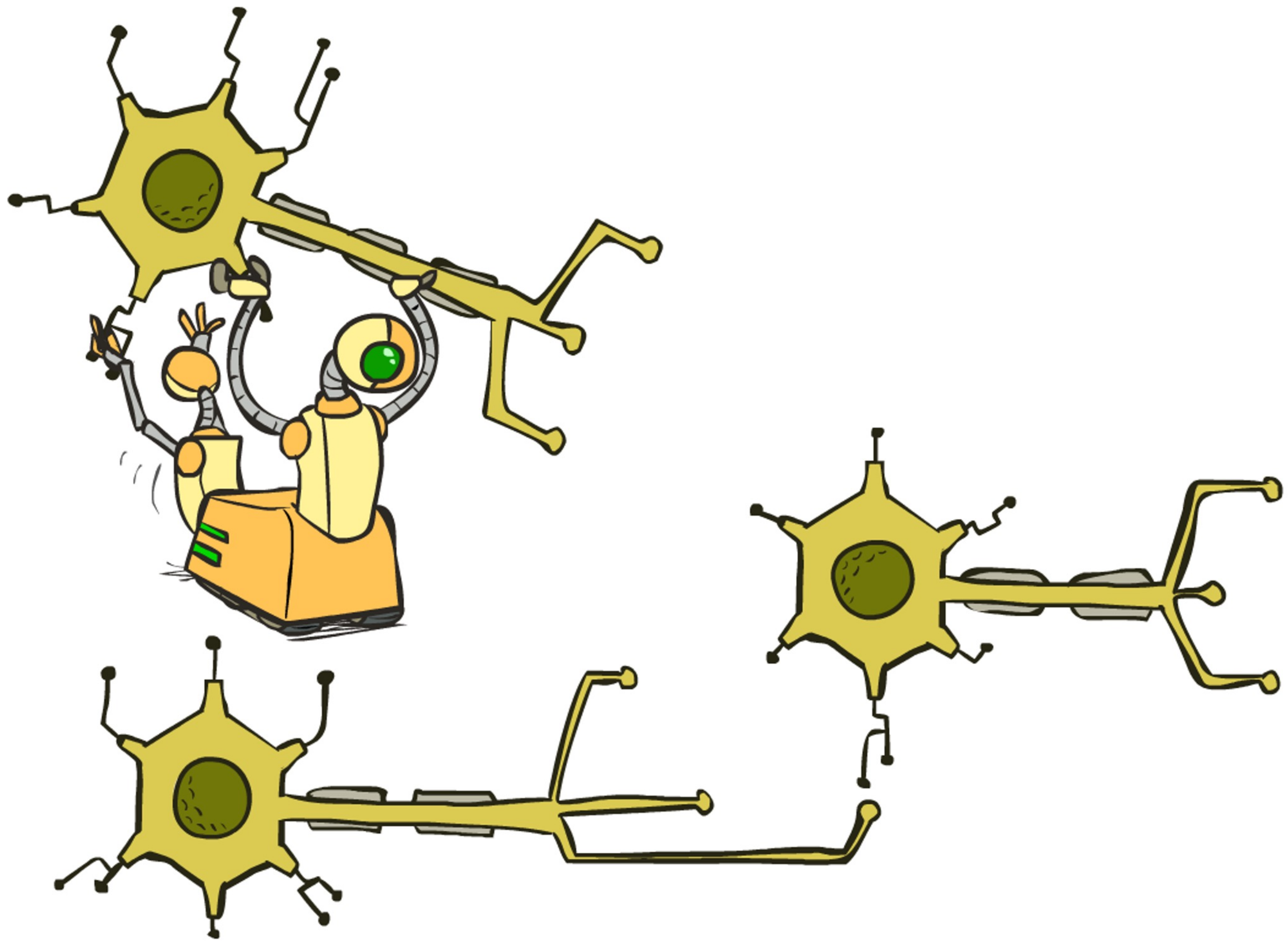
- init w
- for iter = 1, 2, …
  - pick random subset of training examples J

$$w \leftarrow w + \alpha * \sum_{j \in J} \nabla \log P(y^{(j)}|x^{(j)};w)$$

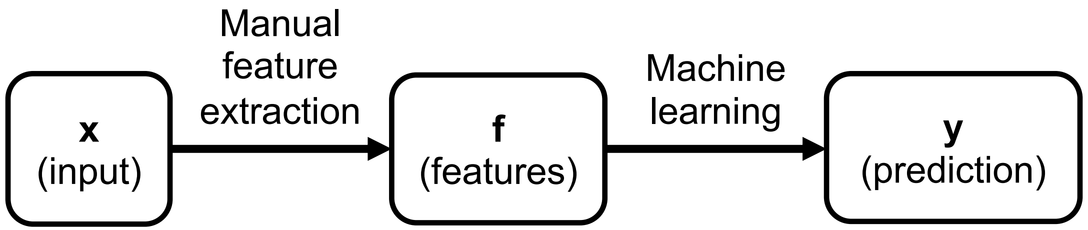
How about computing all the derivatives?

- We'll talk about that once we covered neural networks, which are a generalization of logistic regression

Neural Networks

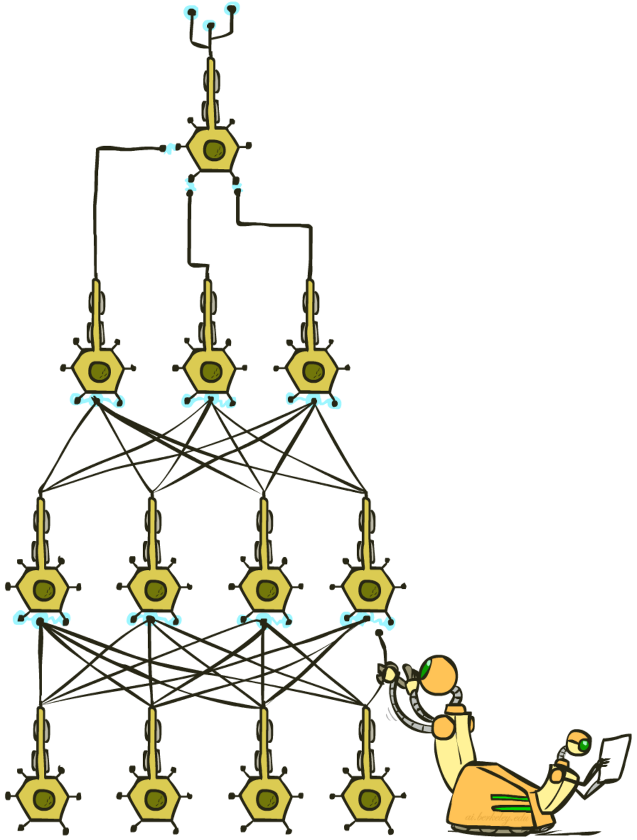


Manual Feature Design vs. Deep Learning

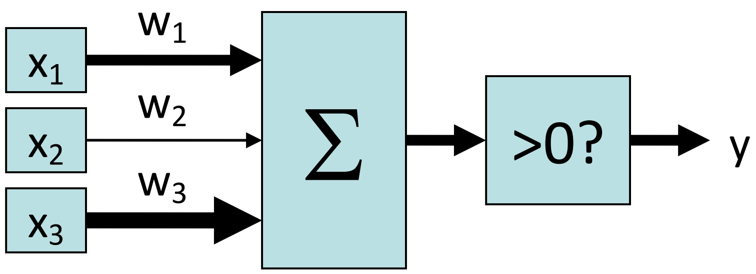


- Manual feature design requires:
  - Domain-specific expertise
  - Domain-specific effort

- What if we could learn the features, too?
  - **Deep Learning**

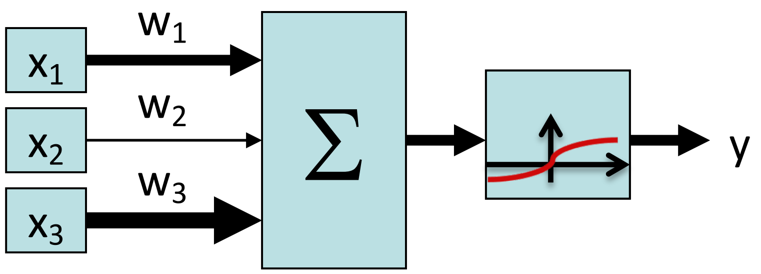


Review: Perceptron



$$y = \begin{cases} 1 & w_1 x_1 + w_2 x_2 + w_3 x_3 > 0 \\ 0 & \text{otherwise} \end{cases}$$

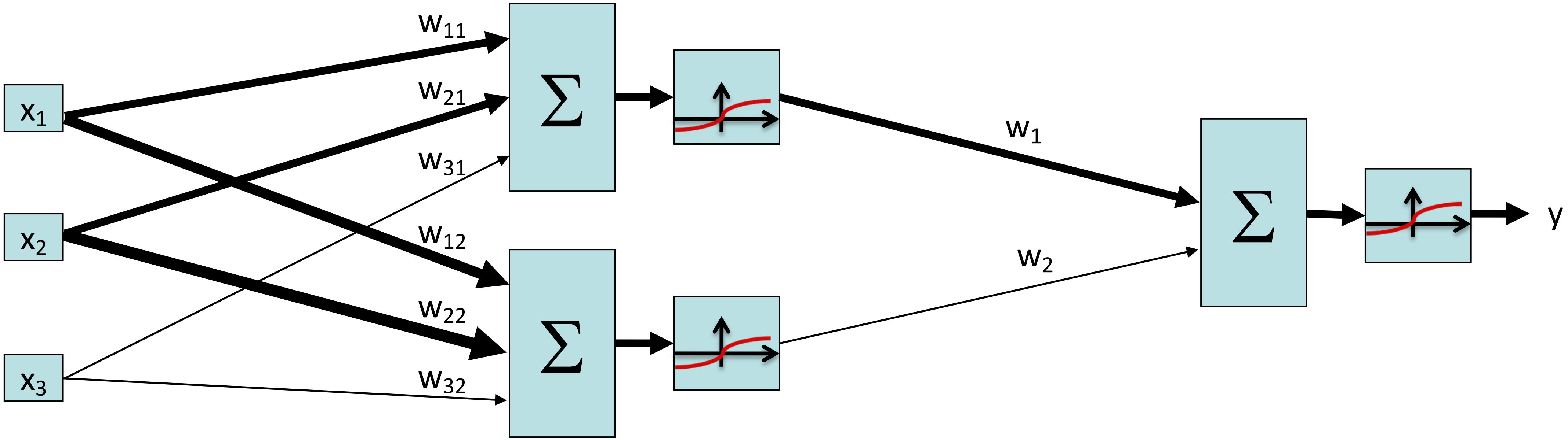
Review: Perceptron with Sigmoid Activation



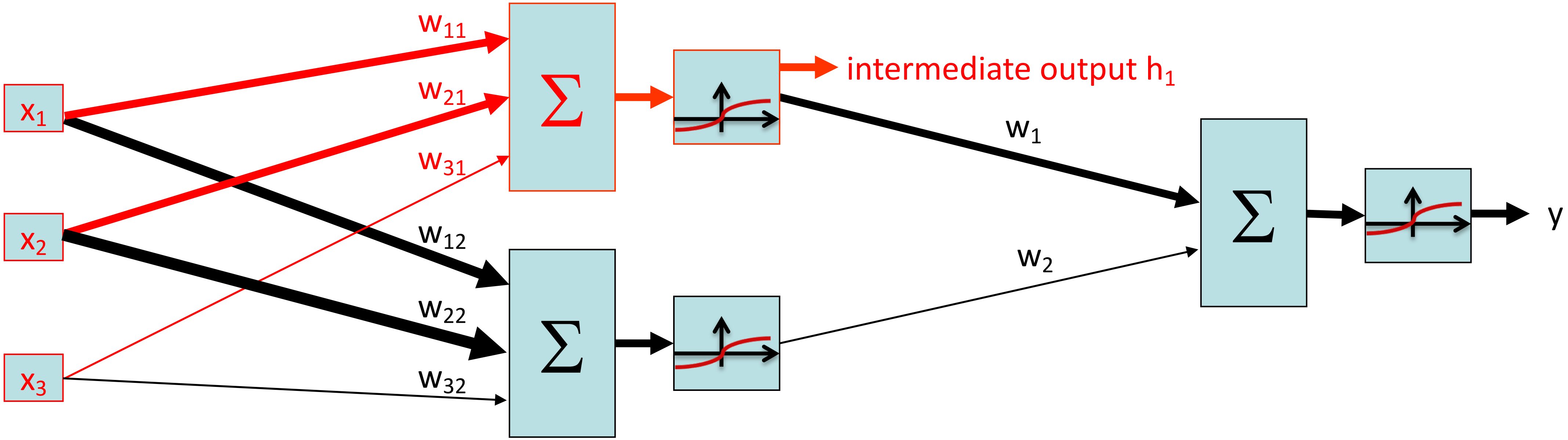
$$y = \phi(w_1 x_1 + w_2 x_2 + w_3 x_3)$$

$$= \frac{1}{1 + e^{-(w_1 x_1 + w_2 x_2 + w_3 x_3)}}$$

2-Layer, 2-Neuron Neural Network

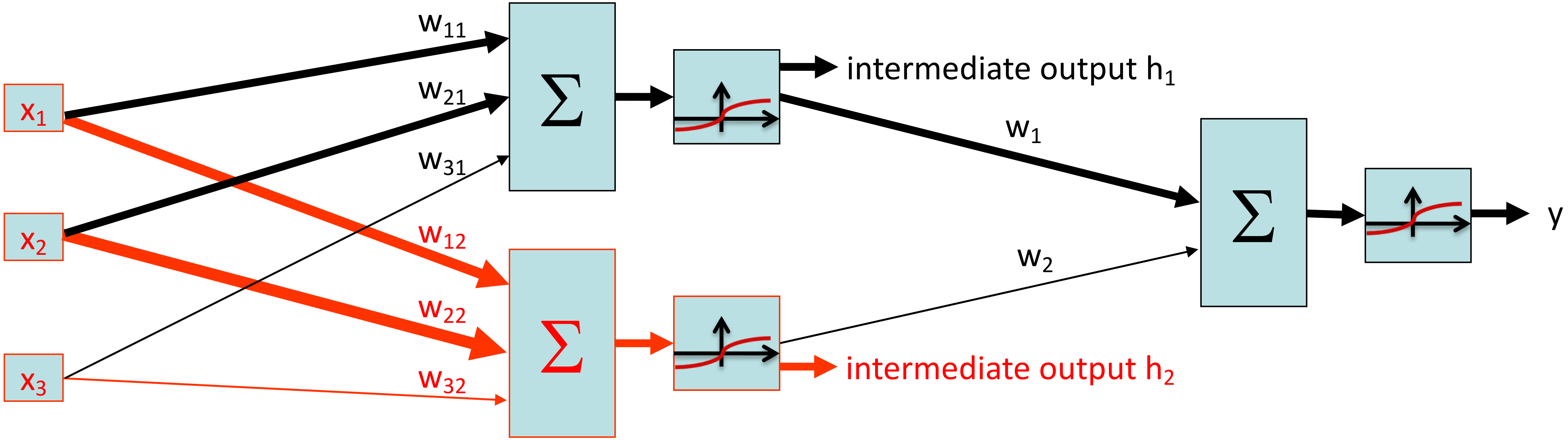


2-Layer, 2-Neuron Neural Network



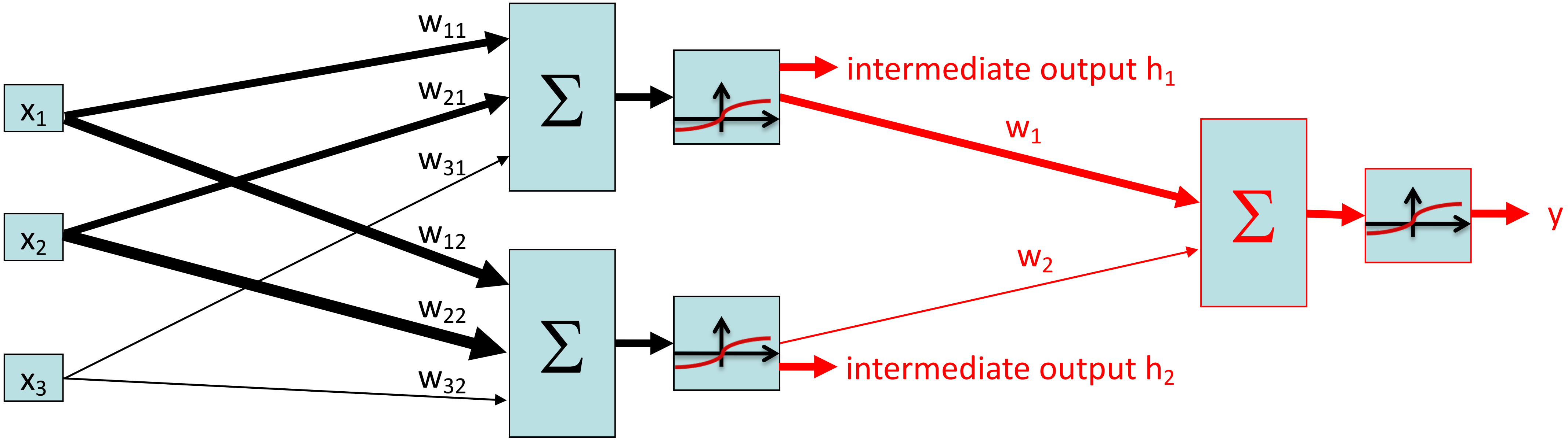
$$\text{intermediate output } h_1 = \phi(w_{11}x_1 + w_{21}x_2 + w_{31}x_3)$$
$$= \frac{1}{1 + e^{-(w_{11}x_1 + w_{21}x_2 + w_{31}x_3)}}$$

2-Layer, 2-Neuron Neural Network



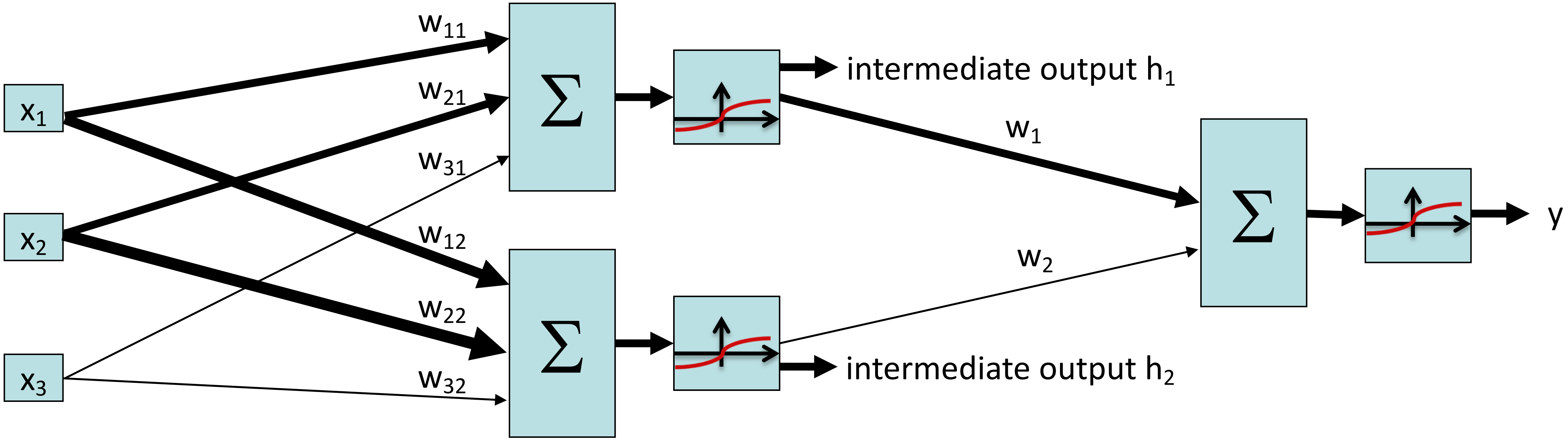
$$\text{intermediate output } h_2 = \phi(w_{12}x_1 + w_{22}x_2 + w_{32}x_3)$$
$$= \frac{1}{1 + e^{-(w_{12}x_1 + w_{22}x_2 + w_{32}x_3)}}$$

2-Layer, 2-Neuron Neural Network



$$y = \phi(w_1 h_1 + w_2 h_2)$$
$$= \frac{1}{1 + e^{-(w_1 h_1 + w_2 h_2)}}$$

2-Layer, 2-Neuron Neural Network



$$y = \phi(w_1 h_1 + w_2 h_2)$$
$$= \phi(w_1 \phi(w_{11} x_1 + w_{21} x_2 + w_{31} x_3) + w_2 \phi(w_{12} x_1 + w_{22} x_2 + w_{32} x_3))$$

2-Layer, 2-Neuron Neural Network

$$y = \phi(w_1 h_1 + w_2 h_2)$$
$$= \phi(w_1 \phi(w_{11}x_1 + w_{21}x_2 + w_{31}x_3) + w_2 \phi(w_{12}x_1 + w_{22}x_2 + w_{32}x_3))$$

The same equation, formatted with matrices:

$$\phi\left(\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}\begin{bmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \\ w_{31} & w_{32} \end{bmatrix}\right)$$
$$= \phi\left(\begin{bmatrix} w_{11}x_1 + w_{21}x_2 + w_{31}x_3 & w_{12}x_1 + w_{22}x_2 + w_{32}x_3 \end{bmatrix}\right)$$
$$= \begin{bmatrix} h_1 & h_2 \end{bmatrix}$$

$$\phi\left(\begin{bmatrix} h_1 & h_2 \end{bmatrix}\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}\right) = \phi\left(w_1 h_1 + w_2 h_2\right) = y$$

The same equation, formatted more compactly by introducing variables representing each matrix:

$$\phi(x \times W_{\text{layer 1}}) = h \qquad \phi(h \times W_{\text{layer 2}}) = y$$

2-Layer, 2-Neuron Neural Network

$$\phi(x \times W_{\text{layer 1}}) = h$$

Shape (1, 3).
Input feature vector.

Shape (3, 2).
Weights to be learned.

Shape (1, 2).
Outputs of layer 1,
inputs to layer 2.

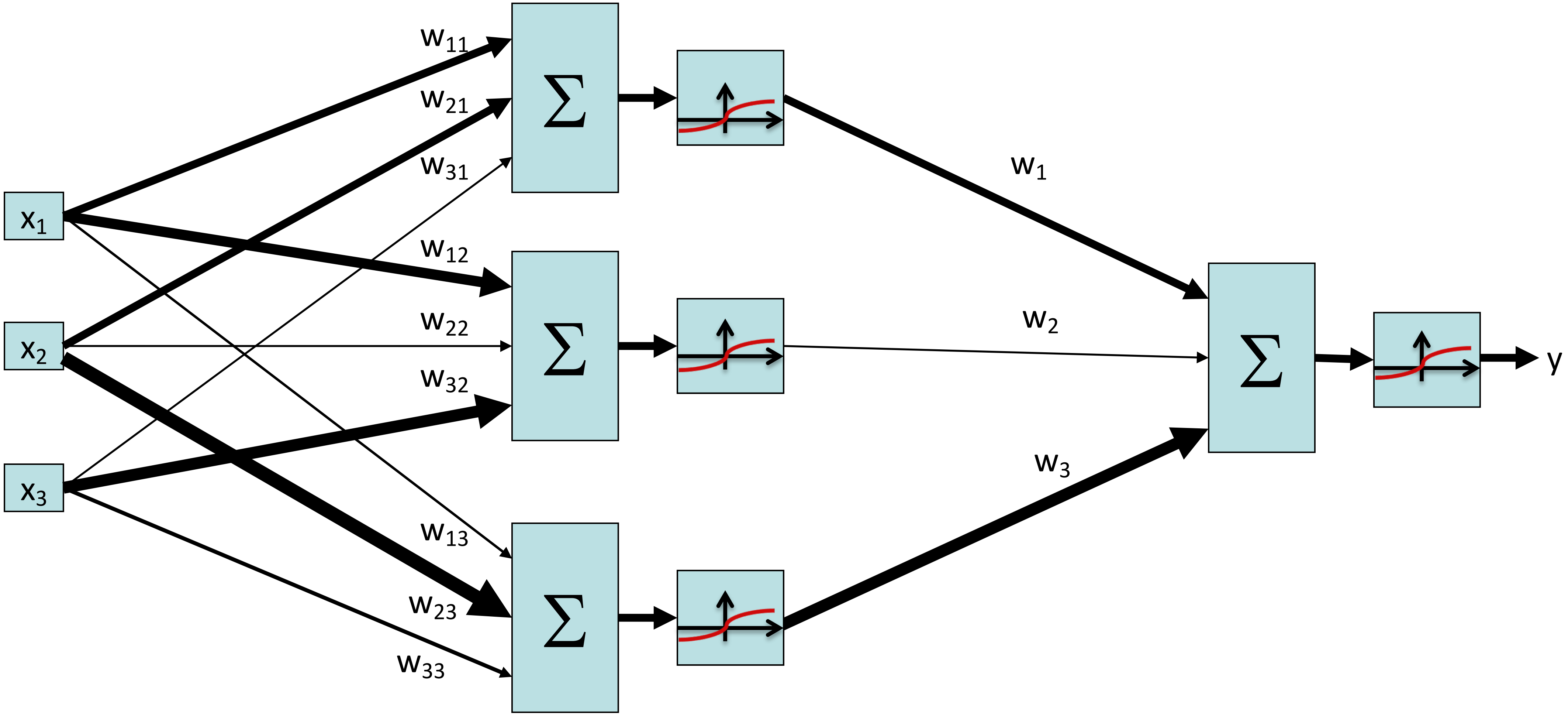
$$\phi(h \times W_{\text{layer 2}}) = y$$

Shape (1, 2).
Outputs of layer 1,
inputs to layer 2.

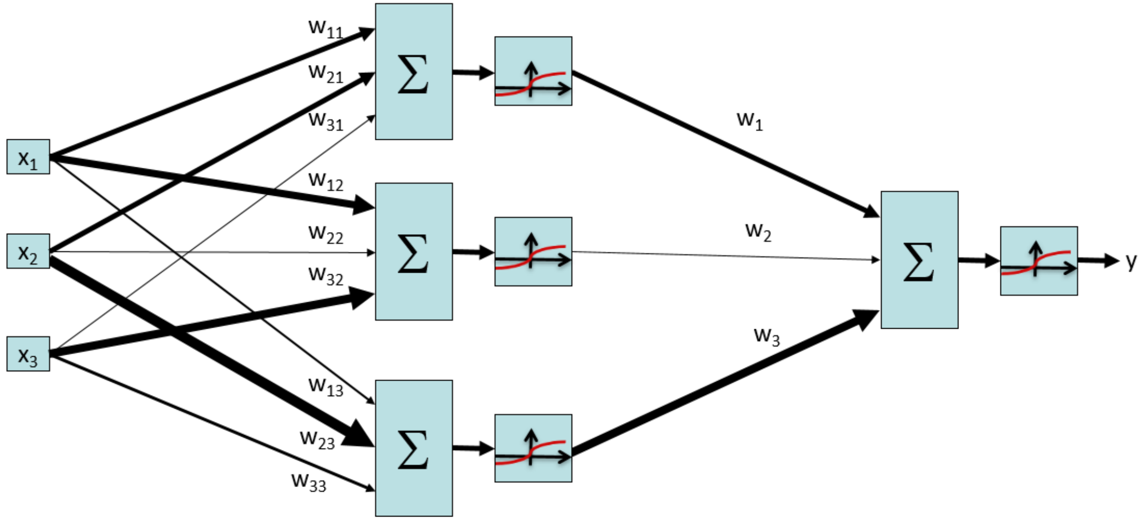
Shape (2, 1).
Weights to be learned.

Shape (1, 1).
Output of network.

2-Layer, 3-Neuron Neural Network



2-Layer, 3-Neuron Neural Network



$$\phi\left(\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}\begin{bmatrix} w_{11} & w_{12} & w_{13} \\ w_{21} & w_{22} & w_{23} \\ w_{31} & w_{32} & w_{33} \end{bmatrix}\right)$$

$$= \phi\left(\begin{bmatrix} w_{11}x_1 + w_{21}x_2 + w_{31}x_3 & w_{12}x_1 + w_{22}x_2 + w_{32}x_3 & w_{13}x_1 + w_{23}x_2 + w_{33}x_3 \end{bmatrix}\right)$$

$$= \begin{bmatrix} h_1 & h_2 & h_3 \end{bmatrix}$$

$$\phi\left(\begin{bmatrix} h_1 & h_2 & h_3 \end{bmatrix}\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}\right) = \phi\left(w_1 h_1 + w_2 h_2 + w_3 h_3\right) = y$$

2-Layer, 3-Neuron Neural Network

$$\phi(x \times W_{\text{layer 1}}) = h$$

Shape (1, 3).
Input feature vector.

Shape (3, 3).
Weights to be learned

Shape (1, 3).
Outputs of layer 1,
inputs to layer 2.

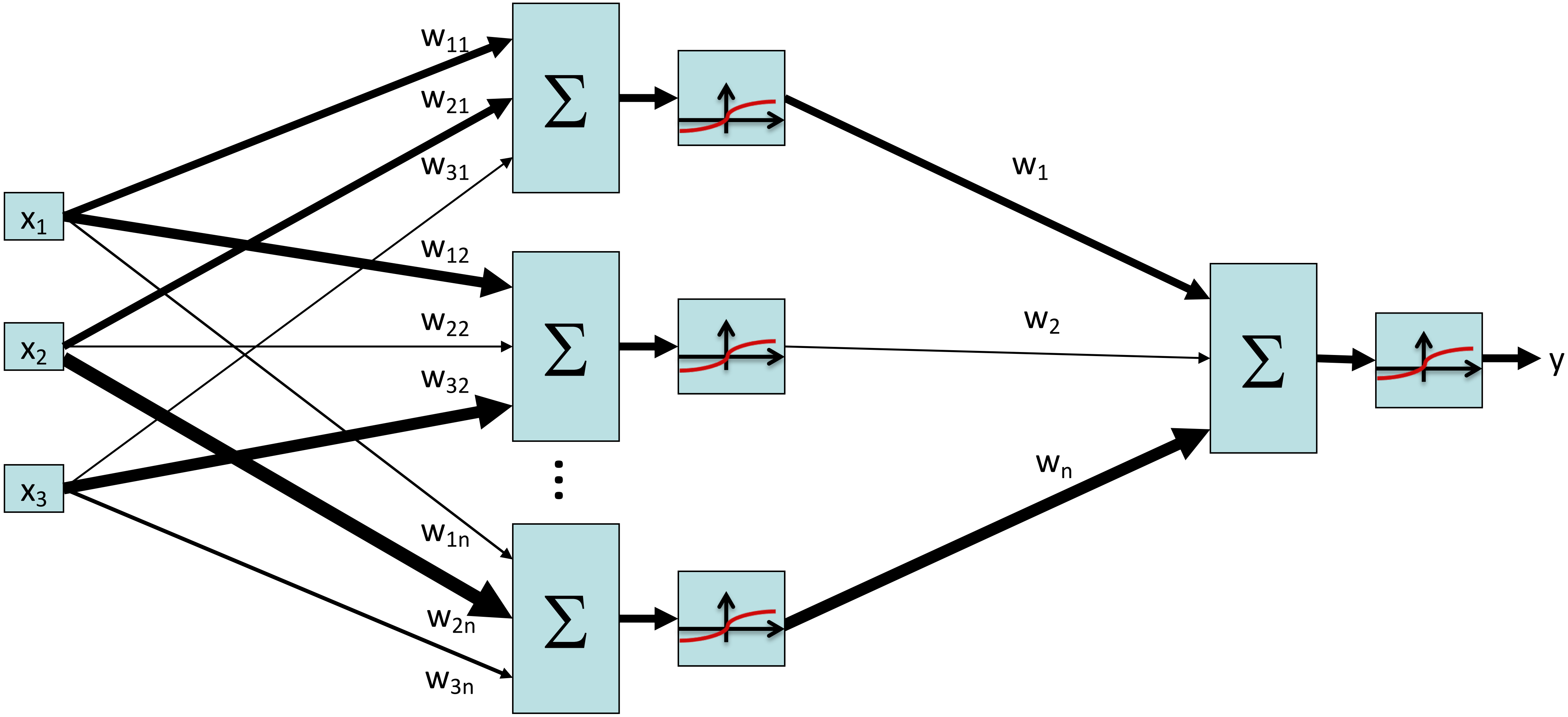
$$\phi(h \times W_{\text{layer 2}}) = y$$

Shape (1, 3).
Outputs of layer 1,
inputs to layer 2.

Shape (3, 1).
Weights to be learned.

Shape (1, 1).
Output of network.

Generalize: Number of hidden neurons



The hidden layer doesn't necessarily need to have 3 neurons; it could have any arbitrary number *n* neurons.

Generalize: n number of hidden neurons

$$\phi(x \times W_{\text{layer 1}}) = h$$

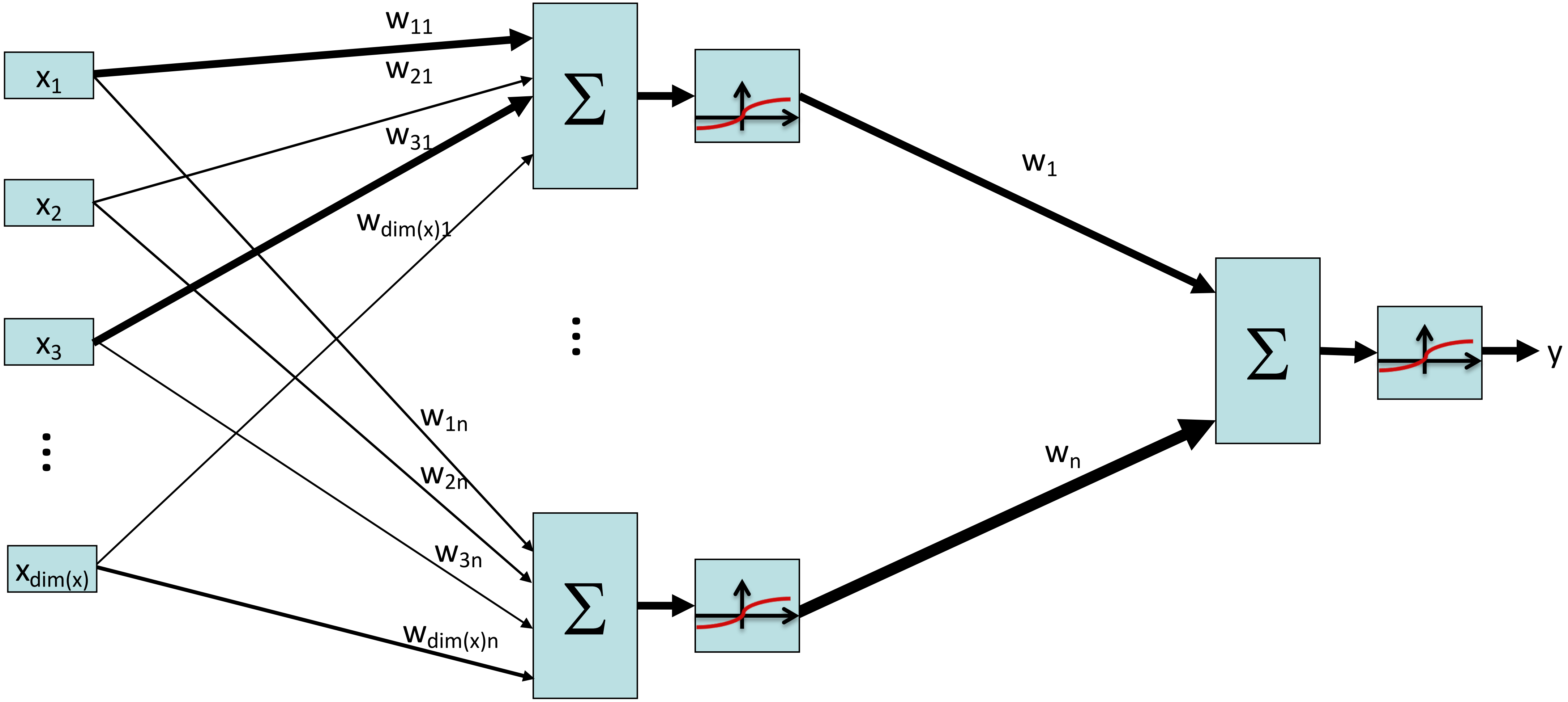
- x: Shape (1, 3). Input feature vector.
- $W_{\text{layer 1}}$: Shape (3, *n*). Weights to be learned
- h: Shape (1, *n*). Outputs of layer 1, inputs to layer 2.

$$\phi(h \times W_{\text{layer 2}}) = y$$

- h: Shape (1, *n*). Outputs of layer 1, inputs to layer 2.
- $W_{\text{layer 2}}$: Shape (*n*, 1). Weights to be learned.
- y: Shape (1, 1). Output of network.

The hidden layer doesn't necessarily need to have 3 neurons; it could have any arbitrary number *n* neurons.

Generalize: Number of input features



The input feature vector doesn't necessarily need to have 3 features; it could have some arbitrary number *dim(x)* of features.

Generalize: Number of input features

$$\phi(x \times W_{\text{layer 1}}) = h$$

Shape (1, *dim(x)*).
Input feature vector.

Shape (*dim(x)*, *n*).
Weights to be learned

Shape (1, *n*).
Outputs of layer 1,
inputs to layer 2.

$$\phi(h \times W_{\text{layer 2}}) = y$$

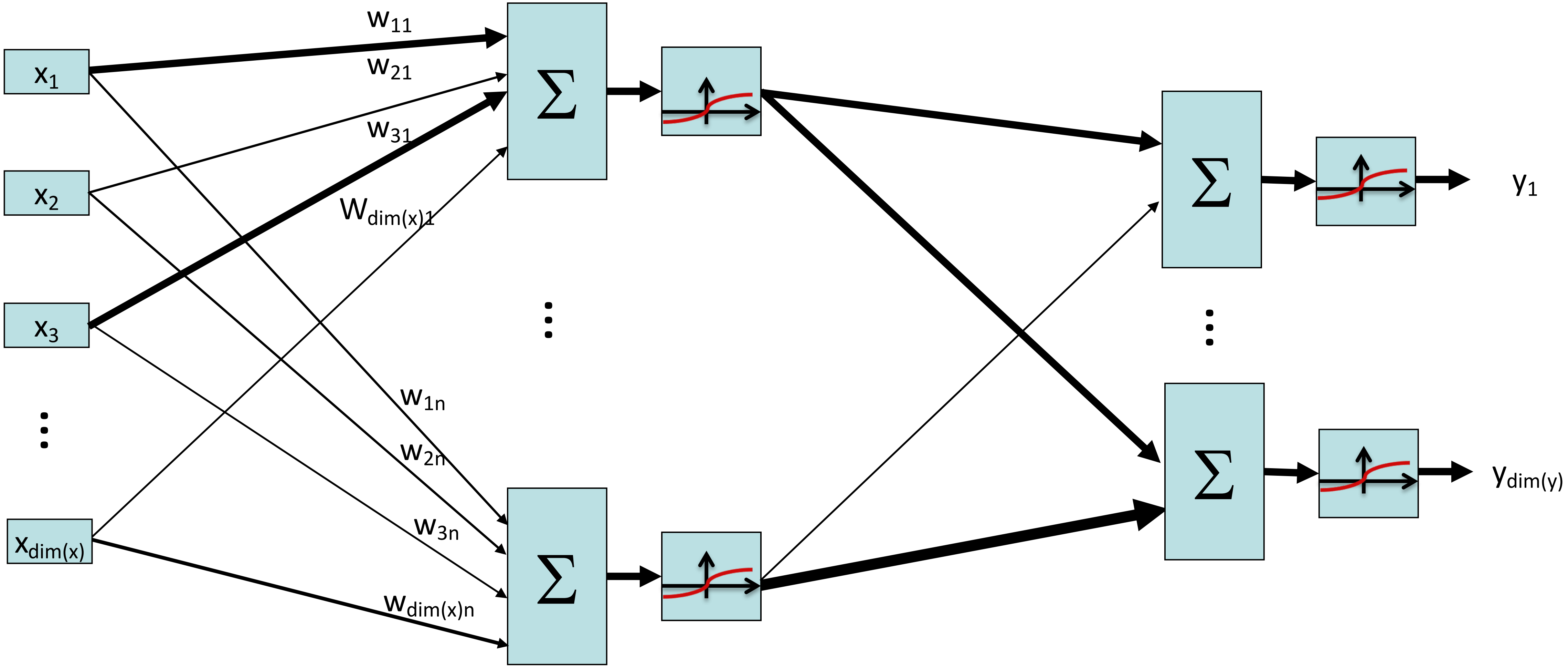
Shape (1, *n*).
Outputs of layer 1,
inputs to layer 2.

Shape (*n*, 1).
Weights to be learned.

Shape (1, 1).
Output of network.

The input feature vector doesn't necessarily need to have 3 features; it could have some arbitrary number *dim(x)* of features.

Generalize: Number of outputs



The output doesn't necessarily need to be just one number; it could be some arbitrary *dim(y)* length vector.

Generalize: Number of input features

$$\phi(x \times W_{\text{layer 1}}) = h$$

Shape (1, *dim(x)*).
Input feature vector.

Shape (*dim(x)*, *n*).
Weights to be learned

Shape (1, *n*).
Outputs of layer 1,
inputs to layer 2.

$$\phi(h \times W_{\text{layer 2}}) = y$$

Shape (1, *n*).
Outputs of layer 1,
inputs to layer 2.

Shape (*n*, *dim(y)*).
Weights to be learned.

Shape (1, *dim(y)*).
Output of network.

The output doesn't necessarily need to be just one number; it could be some arbitrary *dim(y)* length vector.

Generalized 2-Layer Neural Network

$$\phi(x \times W_{\text{layer 1}}) = h$$

Shape (1, *dim(x)*). Input feature vector.

Shape (*dim(x)*, *n*). Weights to be learned

Shape (1, *n*). Outputs of layer 1, inputs to layer 2.

Layer 1 has weight matrix with shape (*dim(x)*, *n*). These are the weights for *n* neurons, each taking *dim(x)* features as input.

This transforms a *dim(x)*-dimensional input vector into an *n*-dimensional output vector.

$$\phi(h \times W_{\text{layer 2}}) = y$$

Shape (1, *n*). Outputs of layer 1, inputs to layer 2.

Shape (*n*, *dim(y)*). Weights to be learned.

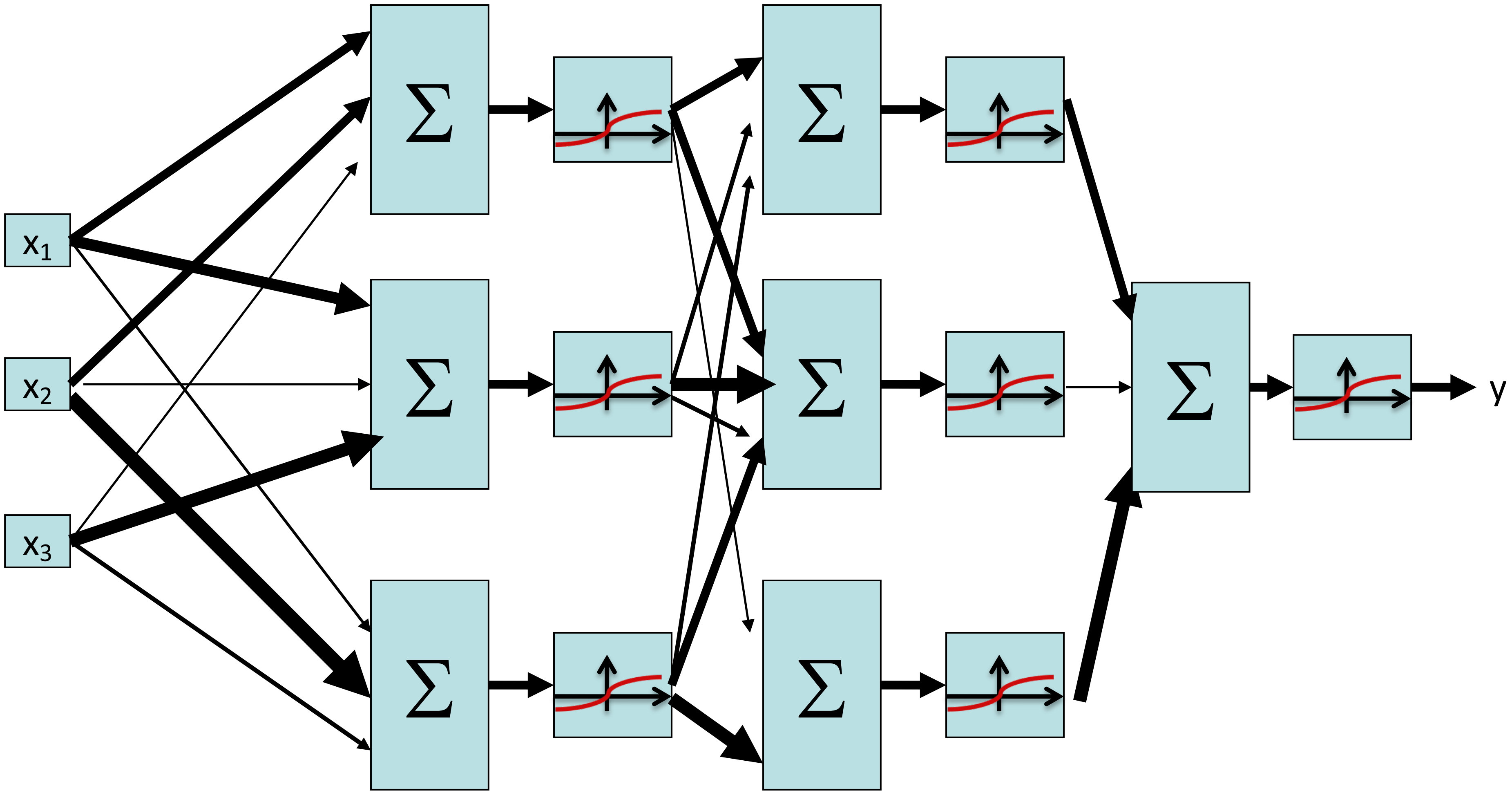
Shape (1, *dim(y)*). Output of network.

Layer 2 has weight matrix with shape (*n*, *dim(y)*). These are the weights for *dim(y)* neurons, each taking *n* features as input.

This transforms an *n*-dimensional input vector into a *dim(y)*-dimensional output vector.

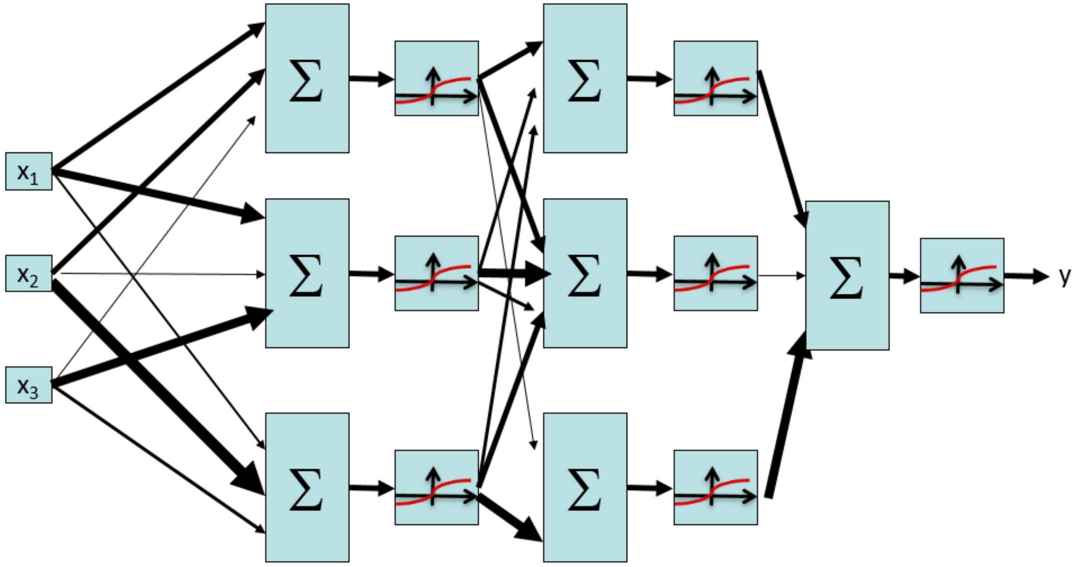
Big idea: The shape of a weight matrix is determined by the dimensions of the input and output of that layer.

3-Layer, 3-Neuron Neural Network



3-Layer, 3-Neuron Neural Network

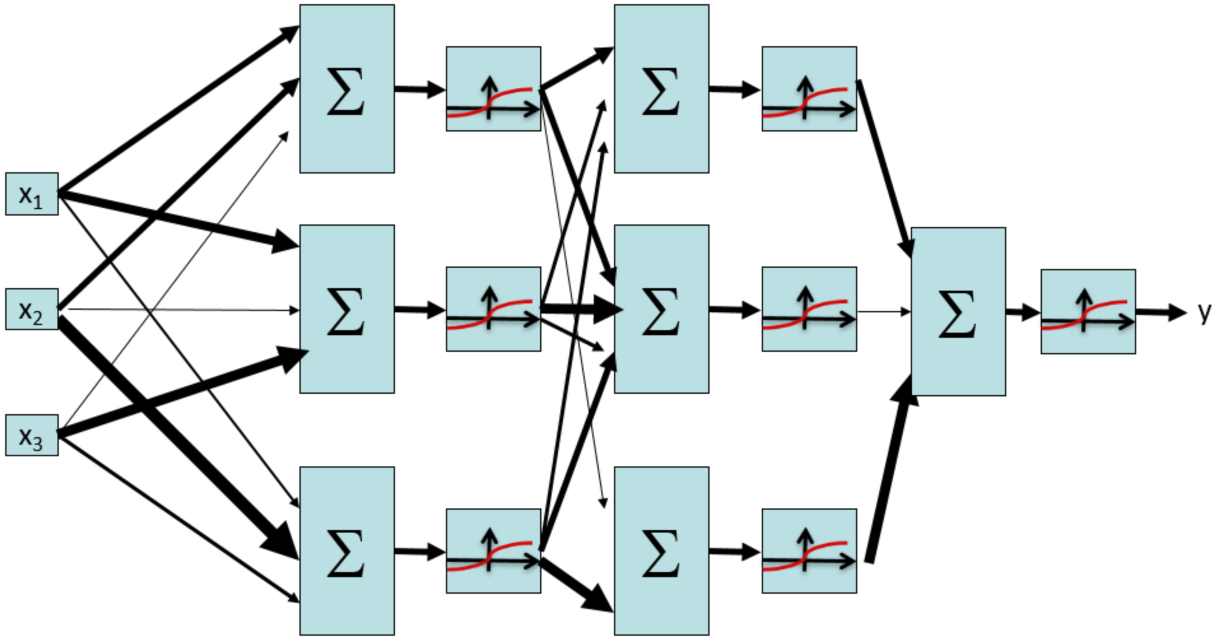
- Layer 1:
  - x has shape (1, 3). Input vector, 3-dimensional.
  - $W_{\text{layer 1}}$ has shape (3, 3). Weights for 3 neurons, each taking in a 3-dimensional input vector.
  - $h_{\text{layer 1}}$ has shape (1, 3). Outputs of the 3 neurons at this layer.
- Layer 2:
  - $h_{\text{layer 1}}$ has shape (1, 3). Outputs of the 3 neurons from the previous layer.
  - $W_{\text{layer 2}}$ has shape (3, 3). Weights for 3 new neurons, each taking in the 3 previous perceptron outputs.
  - $h_{\text{layer 2}}$ has shape (1, 3). Outputs of the 3 new neurons at this layer.
- Layer 3:
  - $h_{\text{layer 2}}$ has shape (1, 3). Outputs from the previous layer.
  - $W_{\text{layer 3}}$ has shape (3, 1). Weights for 1 final neuron, taking in the 3 previous perceptron outputs.
  - y has shape (1, 1). Output of the final neuron.



$$\phi(x \times W_{\text{layer 1}}) = h_{\text{layer 1}}$$
$$\phi(h_{\text{layer 1}} \times W_{\text{layer 2}}) = h_{\text{layer 2}}$$
$$\phi(h_{\text{layer 2}} \times W_{\text{layer 3}}) = y$$

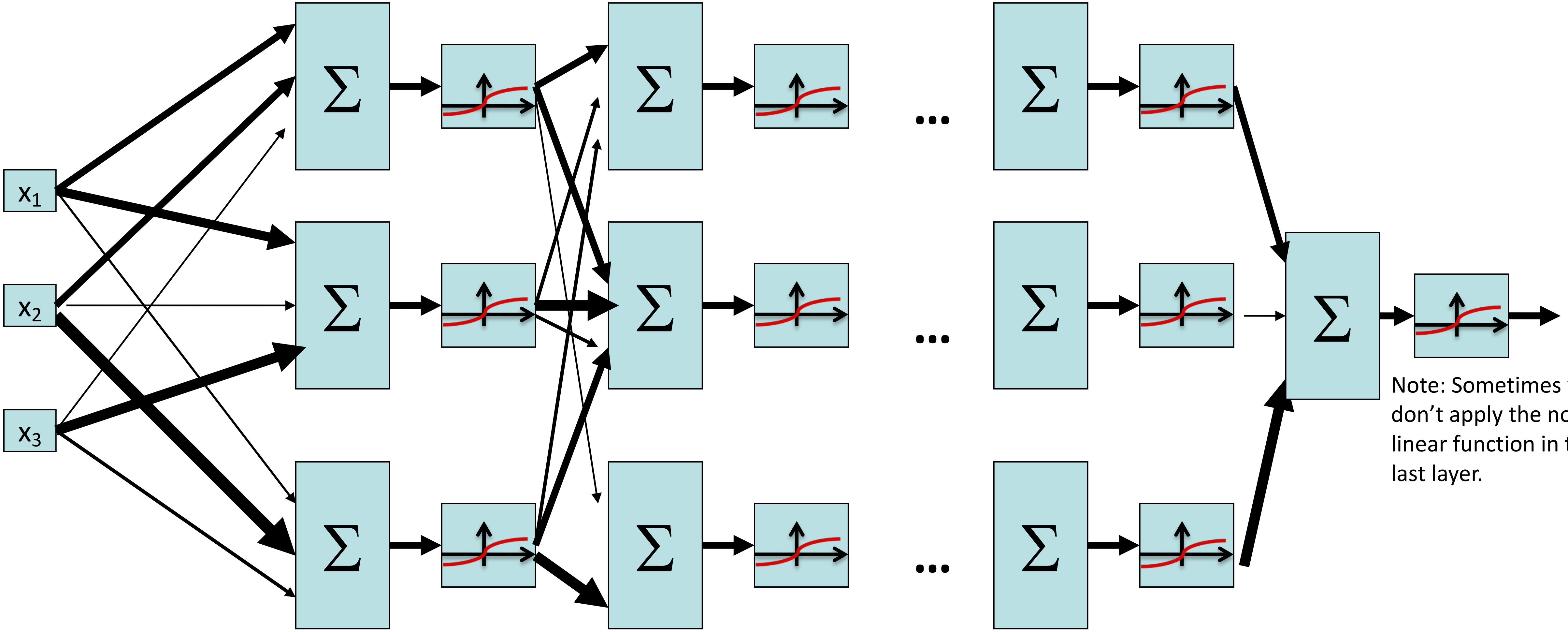
Generalized 3-Layer Neural Network

- Layer 1:
  - x has shape (1, *dim(x)*)
  - $W_{\text{layer 1}}$ has shape (*dim(x)*, *dim(L1)*)
  - $h_{\text{layer 1}}$ has shape (1, *dim(L1)*)
- Layer 2:
  - $h_{\text{layer 1}}$ has shape (1, *dim(L1)*)
  - $W_{\text{layer 2}}$ has shape (*dim(L1)*, *dim(L2)*)
  - $h_{\text{layer 2}}$ has shape (1, *dim(L2)*)
- Layer 3:
  - $h_{\text{layer 2}}$ has shape (1, *dim(L2)*)
  - $W_{\text{layer 3}}$ has shape (*dim(L2)*, *dim(y)*)
  - y has shape (1, *dim(y)*)



$$\phi(x \times W_{\text{layer 1}}) = h_{\text{layer 1}}$$
$$\phi(h_{\text{layer 1}} \times W_{\text{layer 2}}) = h_{\text{layer 2}}$$
$$\phi(h_{\text{layer 2}} \times W_{\text{layer 3}}) = y$$

Multi-Layer Neural Network

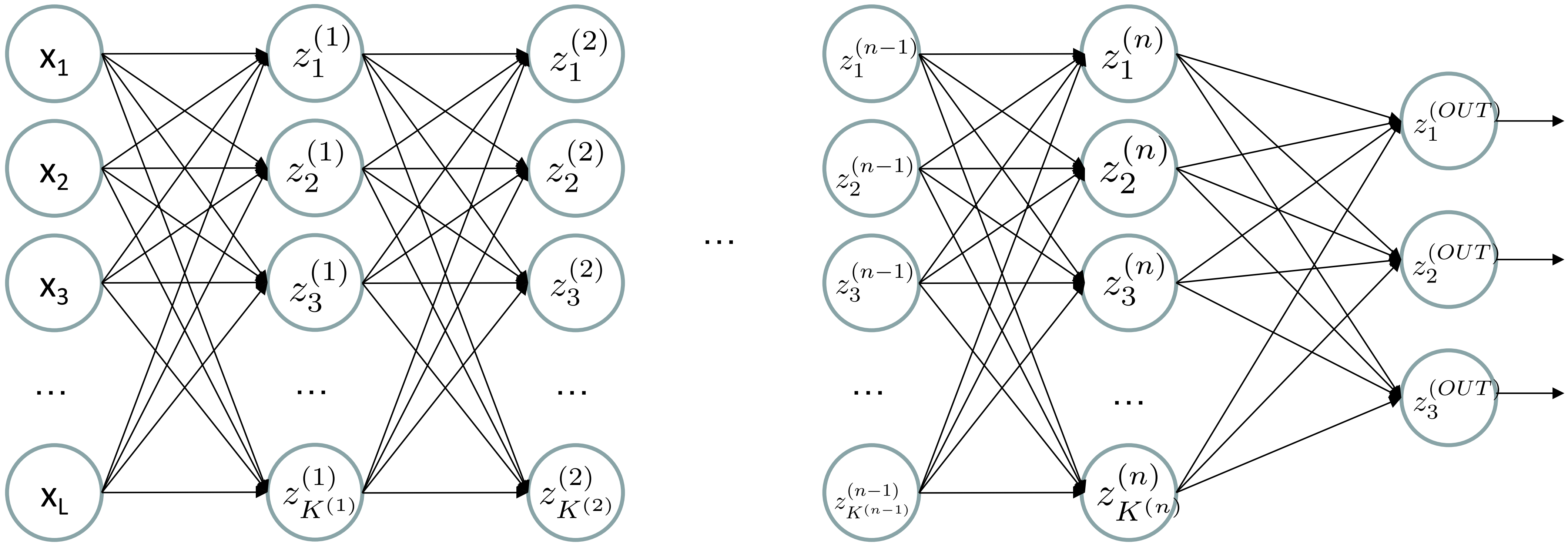


Note: Sometimes we don't apply the non-linear function in the last layer.

Multi-Layer Neural Network

- Input to a layer: some *dim(x)*-dimensional input vector
- Output of a layer: some *dim(y)*-dimensional output vector
  - *dim(y)* is the number of neurons in the layer (1 output per neuron)
- Process of converting input to output:
  - Multiply the (1, *dim(x)*) input vector with a (*dim(x)*, *dim(y)*) weight vector. The result has shape (1, *dim(y)*).
  - Apply some non-linear function (e.g. sigmoid) to the result. The result still has shape (1, *dim(y)*).
- Big idea: Chain layers together
  - The input could come from a previous layer's output
  - The output could be used as the input to the next layer

Deep Neural Network

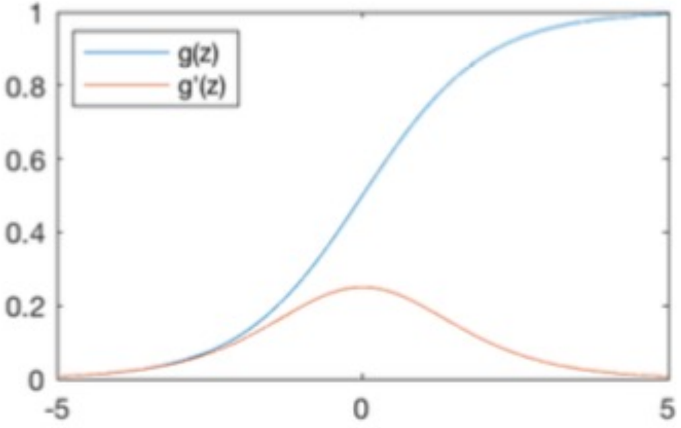


$$z_i^{(k)} = g(\sum_j W_{i,j}^{(k-1,k)} z_j^{(k-1)})$$

g = nonlinear activation function

Common Activation Functions

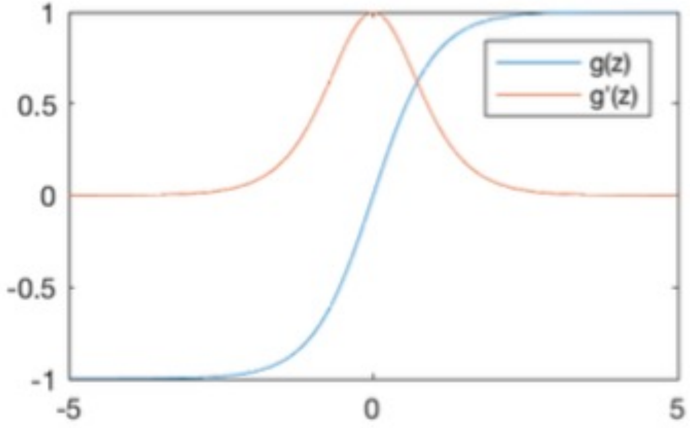
Sigmoid Function



$$g(z) = \frac{1}{1 + e^{-z}}$$

$$g'(z) = g(z)(1 - g(z))$$

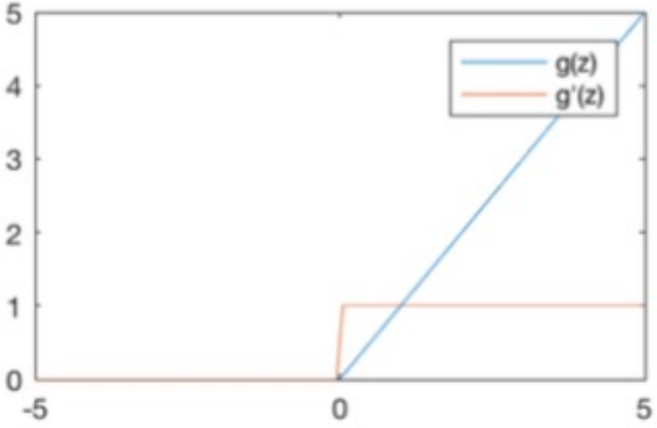
Hyperbolic Tangent



$$g(z) = \frac{e^{z} - e^{-z}}{e^{z} + e^{-z}}$$

$$g'(z) = 1 - g(z)^2$$

Rectified Linear Unit (ReLU)

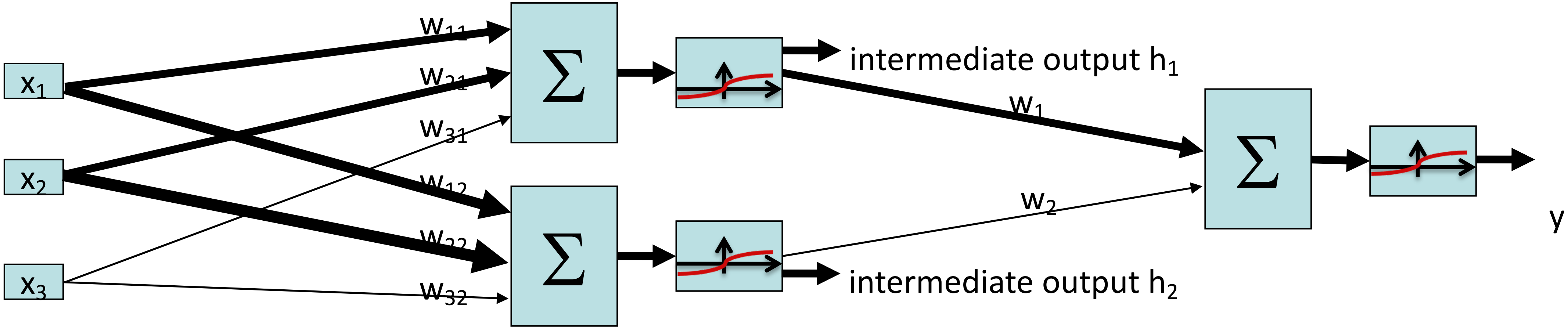


$$g(z) = \max(0, z)$$

$$g'(z) = \begin{cases} 1, & z > 0 \\ 0, & \text{otherwise} \end{cases}$$

[source: MIT 6.S191 introtodeeplearning.com]

Important to use non-linear activation functions



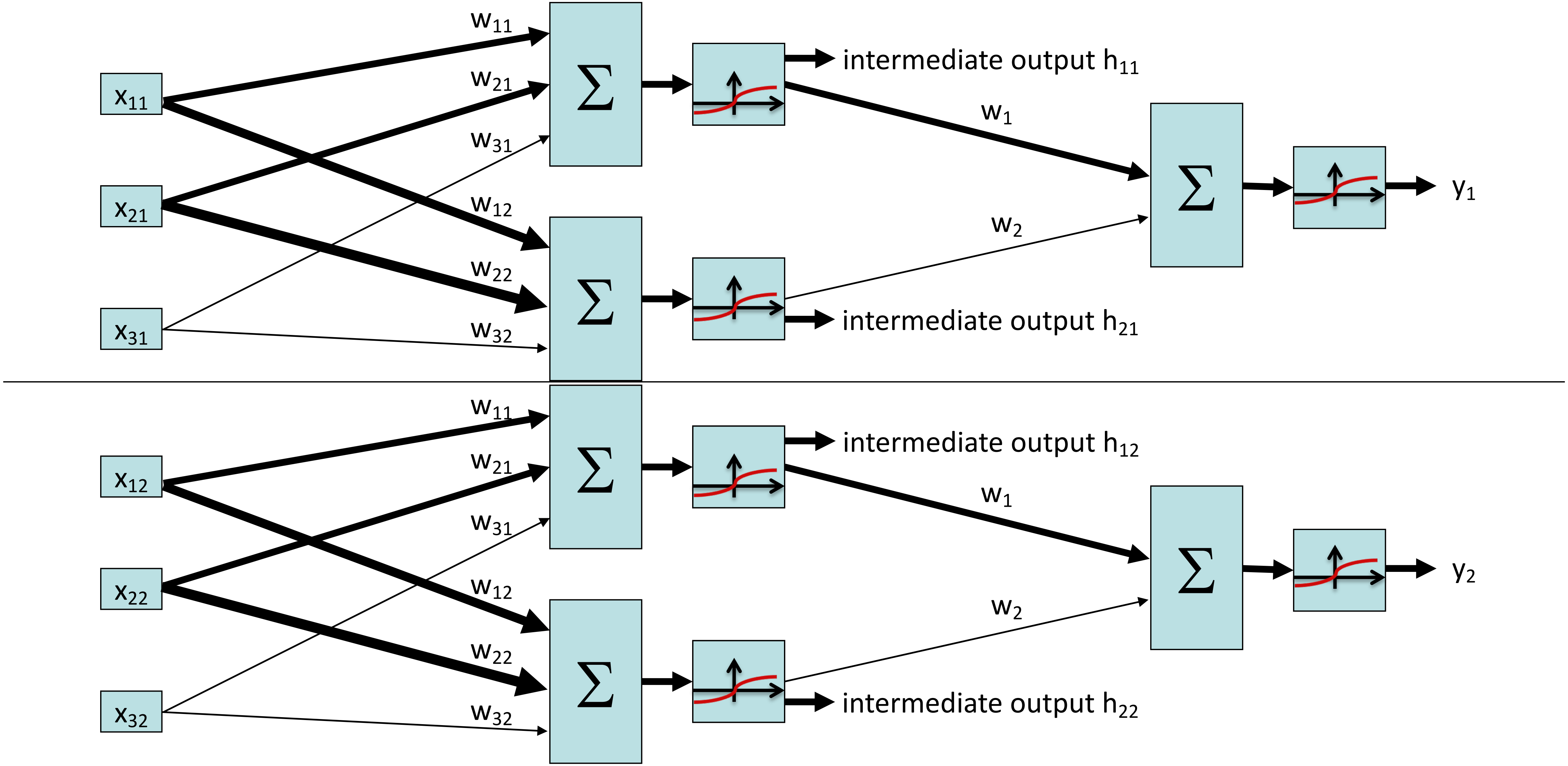
- **With** non-linear activation ϕ for intermediate output:

$$y = \phi(w_1 h_1 + w_2 h_2)$$
$$= \phi(w_1 \phi(w_{11} x_1 + w_{21} x_2 + w_{31} x_3) + w_2 \phi(w_{12} x_1 + w_{22} x_2 + w_{32} x_3))$$

- **Without** intermediate activations ϕ:

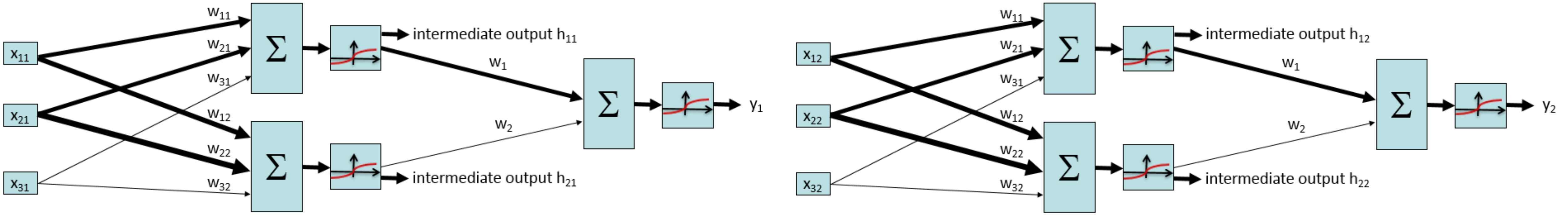
$$y = \phi(w_1(w_{11}x_1 + w_{21}x_2 + w_{31}x_3) + w_2(w_{12}x_1 + w_{22}x_2 + w_{32}x_3))$$
$$= \phi((w_1 w_{11} + w_2 w_{12})x_1 + (w_1 w_{21} + w_2 w_{22})x_2 + (w_1 w_{31} + w_2 w_{32})x_3)$$
$$= \phi(ax_1 + bx_2 + cx_3)$$ ← same as not including a hidden layer!

Batch Sizes



w11
w21
w31
w12
w22
w32
x11
x21
x31
intermediate output h11
intermediate output h21
w1
w2
y1
w11
w21
w31
w12
w22
w32
x12
x22
x32
intermediate output h12
intermediate output h22
w1
w2
y2

Batch Sizes



$$
\begin{aligned}
y_1 &= \phi(w_1 h_{11} + w_2 h_{12}) \\
&= \phi(w_1 \phi(w_{11} x_{11} + w_{21} x_{12} + w_{31} x_{13}) + w_2 \phi(w_{12} x_{11} + w_{22} x_{12} + w_{32} x_{13})) \\
y_2 &= \phi(w_1 h_{21} + w_2 h_{22}) \\
&= \phi(w_1 \phi(w_{11} x_{21} + w_{21} x_{22} + w_{31} x_{23}) + w_2 \phi(w_{12} x_{21} + w_{22} x_{22} + w_{32} x_{23}))
\end{aligned}
$$

We’re not changing the architecture; we’re just running the 2-neuron, 2-layer network twice to classify 2 inputs.

Batch Sizes

$$
\begin{aligned}
y_1 &= \phi(w_1 h_{11} + w_2 h_{12}) \\
&= \phi(w_1 \phi(w_{11} x_{11} + w_{21} x_{12} + w_{31} x_{13}) + w_2 \phi(w_{12} x_{11} + w_{22} x_{12} + w_{32} x_{13})) \\
y_2 &= \phi(w_1 h_{21} + w_2 h_{22}) \\
&= \phi(w_1 \phi(w_{11} x_{21} + w_{21} x_{22} + w_{31} x_{23}) + w_2 \phi(w_{12} x_{21} + w_{22} x_{22} + w_{32} x_{23}))
\end{aligned}
$$

Rewriting in matrix form:

$$
\begin{aligned}
& \phi\left(\begin{bmatrix} x_{11} & x_{21} & x_{31} \\ x_{12} & x_{22} & x_{32} \end{bmatrix}\begin{bmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \\ w_{31} & w_{32} \end{bmatrix}\right) \\
= \ & \phi\left(\begin{bmatrix} w_{11} x_{11} + w_{21} x_{21} + w_{31} x_{31} & w_{12} x_{11} + w_{22} x_{21} + w_{32} x_{31} \\ w_{11} x_{12} + w_{21} x_{22} + w_{31} x_{32} & w_{12} x_{12} + w_{22} x_{22} + w_{32} x_{32} \end{bmatrix}\right) \\
= \ & \begin{bmatrix} h_{11} & h_{21} \\ h_{12} & h_{22} \end{bmatrix}
\end{aligned}
$$

$$
\phi\left(\begin{bmatrix} h_{11} & h_{21} \\ h_{12} & h_{22} \end{bmatrix}\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}\right) = \phi\left(\begin{bmatrix} w_1 h_{11} + w_2 h_{21} \\ w_1 h_{12} + w_2 h_{22} \end{bmatrix}\right) = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}
$$

Batch Sizes

$$\phi(x \times W_{\text{layer 1}}) = h$$

- x: Shape (*batch*, *dim(x)*). Input feature vector.
- $W_{\text{layer 1}}$: Shape (*dim(x)*, *n*). Weights to be learned
- h: Shape (*batch*, *n*). Outputs of layer 1, inputs to layer 2.

$$\phi(h \times W_{\text{layer 2}}) = y$$

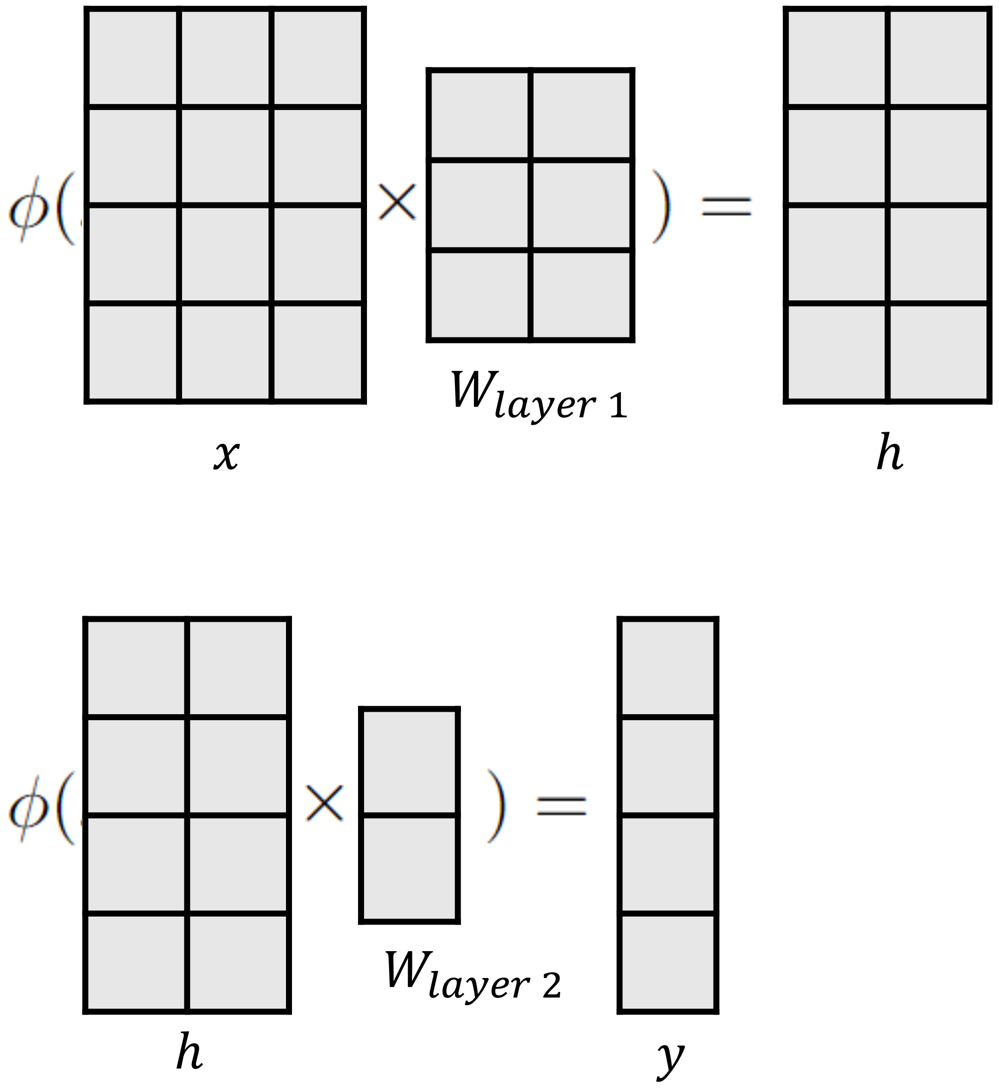
- h: Shape (*batch*, *n*). Outputs of layer 1, inputs to layer 2.
- $W_{\text{layer 2}}$: Shape (*n*, *dim(y)*). Weights to be learned.
- y: Shape (*batch*, *dim(y)*). Output of network.

Big idea: We can “stack” inputs together to classify multiple inputs at once. The result is multiple outputs “stacked” together.

Multi-Layer Network, with Batches

- Input to a layer: *batch* different *dim(x)*-dimensional input vectors
- Output of a layer: *batch* different *dim(y)*-dimensional output vectors
  - *dim(y)* is the number of neurons in the layer (1 output per neuron)
- Process of converting input to output:
  - Multiply the (*batch*, *dim(x)*) input matrix with a (*dim(x)*, *dim(y)*) weight vector. The result has shape (*batch*, *dim(y)*).
  - Apply some non-linear function (e.g. sigmoid) to the result. The result still has shape (*batch*, *dim(y)*).
- Big idea: Stack inputs/outputs to batch them
  - The multiplication by weights and non-linear function will be applied to each row (data point in the batch) separately.

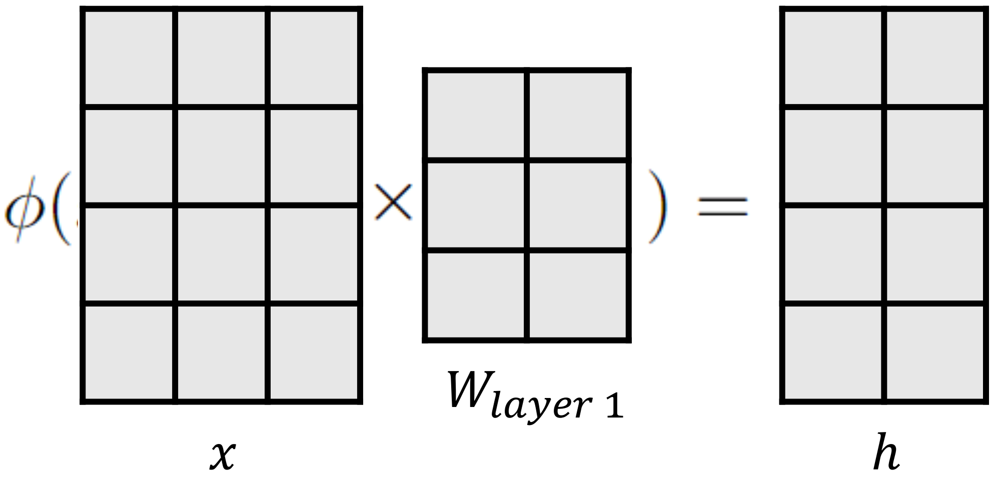
Quiz: Sizes of neural networks

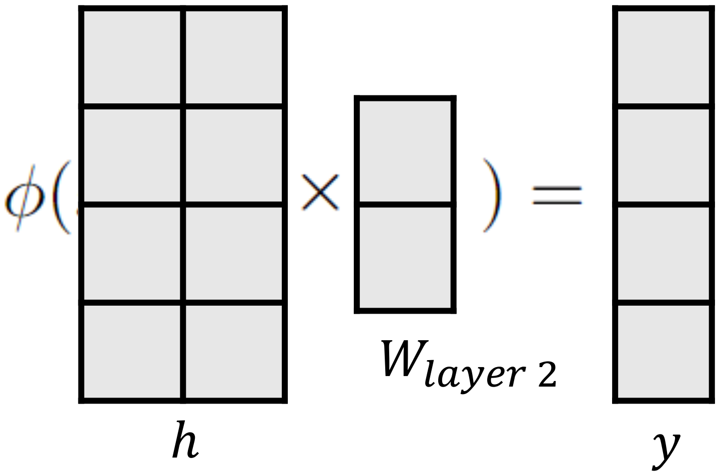


We have a neural network with the matrices drawn.

1. How many layers are in the network?
2. How many input dimensions dim(x)?
3. How many hidden neurons n?
4. How many output dimensions dim(y)?
5. What is the batch size?

Quiz: Sizes of neural networks





We have a neural network with the matrices drawn.

1. How many layers are in the network?
   2
2. How many input dimensions dim(x)?
   3
3. How many hidden neurons n?
   2
4. How many output dimensions dim(y)?
   1
5. What is the batch size?
   4

Next Time: Training Neural Networks & Applications