CS 188: Artificial Intelligence
Spring 2006

Lecture 18: HMMs
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Today

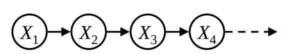
- Last time:
  - Bayes nets
  - Answering queries with variable elimination
- Today:
  - Reasoning over time
  - Markov processes
  - Hidden Markov models

Reasoning over Time

- Often, we want to reason about a sequence of observations
  - Speech recognition
  - Robot localization
  - User attention
- Need to introduce time into our models
- Basic approach: hidden Markov models (HMMs)
- More general: dynamic Bayes' nets

Markov Models

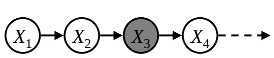
- A Markov model is a chain-structured BN
  - Each node is identically distributed (stationarity)
  - Value of X at a given time is called the state
  - As a BN:



$P(X_1)$ $\qquad$ $P(X|X_{-1})$

  - Parameters: called transition probabilities, specify how the state evolves over time (also, initial probs)

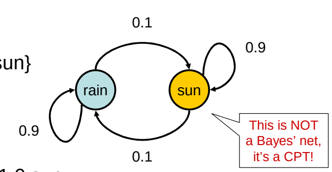
Conditional Independence



- Basic conditional independence:
  - Past and future independent of the present
  - Each time step only depends on the previous
  - This is called the (first order) Markov property

Example

- Weather:
  - States: X = {rain, sun}
  - Transitions:



  - Initial distribution: 1.0 sun
  - What's the probability distribution after one step?

$$P(X_2 = \text{sun}) = P(X_2 = \text{sun}|X_1 = \text{sun})P(X_1 = \text{sun}) + P(X_2 = \text{sun}|X_1 = \text{rain})P(X_1 = \text{rain})$$

$$0.9 \cdot 1.0 + 0.1 \cdot 0.0 = 0.9$$

Mini-Forward Algorithm

- Question: probability of being in state x at time t?
- Slow answer:
  - Enumerate all sequences of length t which end in s
  - Add up their probabilities
- Better answer: cached incremental belief update

$$P(x_t) = \sum_{x_{t-1}} P(x_t|x_{t-1})P(x_{t-1})$$

$$P(x_1) = \text{known}$$

Forward simulation

Example

- From initial observation of sun

$$\left\langle \begin{matrix} 1.0 \\ 0.0 \end{matrix} \right\rangle \quad \left\langle \begin{matrix} 0.9 \\ 0.1 \end{matrix} \right\rangle \quad \left\langle \begin{matrix} 0.82 \\ 0.18 \end{matrix} \right\rangle \quad \Rightarrow \quad \left\langle \begin{matrix} 0.5 \\ 0.5 \end{matrix} \right\rangle$$

$\mathrm{P}(X_1)$ $\mathrm{P}(X_2)$ $\mathrm{P}(X_3)$ $\mathrm{P}(X_\infty)$

- From initial observation of rain

$$\left\langle \begin{matrix} 0.0 \\ 1.0 \end{matrix} \right\rangle \quad \left\langle \begin{matrix} 0.1 \\ 0.9 \end{matrix} \right\rangle \quad \left\langle \begin{matrix} 0.18 \\ 0.82 \end{matrix} \right\rangle \quad \Rightarrow \quad \left\langle \begin{matrix} 0.5 \\ 0.5 \end{matrix} \right\rangle$$

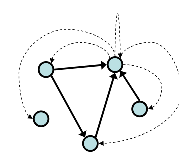
$\mathrm{P}(X_1)$ $\mathrm{P}(X_2)$ $\mathrm{P}(X_3)$ $\mathrm{P}(X_\infty)$

Stationary Distributions

- If we simulate the chain long enough:
  - What happens?
  - Uncertainty accumulates
  - Eventually, we have no idea what the state is!
- Stationary distributions:
  - For most chains, the distribution we end up in is independent of the initial distribution
  - Called the stationary distribution of the chain
  - Usually, can only predict a short time out

Web Link Analysis

- PageRank over a web graph
  - Each web page is a state
  - Initial distribution: uniform over pages
  - Transitions:
    - With prob. c, uniform jump to a random page (solid lines)
    - With prob. 1-c, follow a random outlink (dotted lines)



- Stationary distribution
  - Will spend more time on highly reachable pages
  - E.g. many ways to get to the Acrobat Reader download page!
  - Somewhat robust to link spam
  - Google 1.0 returned pages containing your keywords in decreasing rank, now all search engines use link analysis along with many other factors

Most Likely Explanation

- Question: most likely sequence ending in x at t?
  - E.g. if sun on day 4, what's the most likely sequence?
  - Intuitively: probably sun all four days
- Slow answer: enumerate and score

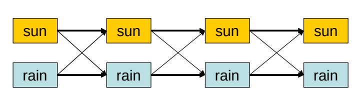
$$P(X_1 = sun)P(X_2 = sun|X_1 = sun)P(X_3 = sun|X_2 = sun)P(X_4 = sun|X_3 = sun)$$

$$P(X_1 = sun)P(X_2 = rain|X_1 = sun)P(X_3 = sun|X_2 = rain)P(X_4 = sun|X_3 = sun)$$

$$\vdots$$

Mini-Viterbi Algorithm

- Better answer: cached incremental updates

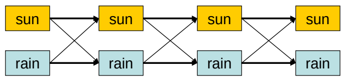


- Define: $m_t[x] = \max_{x_{1:t-1}} P(x_{1:t-1}, x)$

$$a_t[x] = \arg\max_{x_{1:t-1}} P(x_{1:t-1}, x)$$

- Read best sequence off of m and a vectors

Mini-Viterbi



$$m_t[x] = \max_{x_{1:t-1}} P(x_{1:t-1}, x)$$

$$= \max_{x_{1:t-1}} P(x_{1:t-1}) P(x|x_{t-1})$$

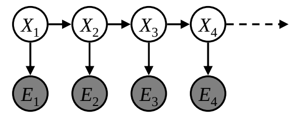
$$= \max_{x_{t-1}} P(x_t|x_{t-1}) \max_{x_{1:t-2}} P(x_{1:t-1})$$

$$= \max_{x_{t-1}} P(x_t|x_{t-1}) m_{t-1}[x]$$

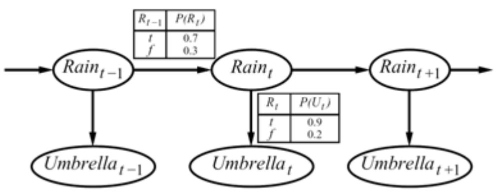
$$m_1[x] = P(x_1)$$

Hidden Markov Models

- Markov chains not so useful for most agents
  - Eventually you don't know anything anymore
  - Need observations to update your beliefs
- Hidden Markov models (HMMs)
  - Underlying Markov chain over states S
  - You observe outputs (effects) at each time step
  - As a Bayes' net:



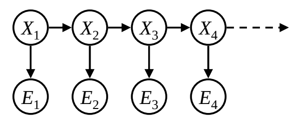
Example



- An HMM is
  - Initial distribution: $P(X_1)$
  - Transitions: $P(X|X_{-1})$
  - Emissions: $P(E|X)$

Conditional Independence

- HMMs have two important independence properties:
  - Markov hidden process, future depends on past via the present
  - Current observation independent of all else given current state



- Quiz: does this mean that observations are independent given no evidence? Why? [No, correlated by the hidden state]

Forward Algorithm

- Can ask the same questions for HMMs as Markov chains
- Given current belief state, how to update with evidence?
  - This is called monitoring or filtering
- Formally, we want: $P(X_t = x_t|e_{1:t})$

$$P(x_t|e_{1:t}) \propto P(x_t, e_{1:t})$$

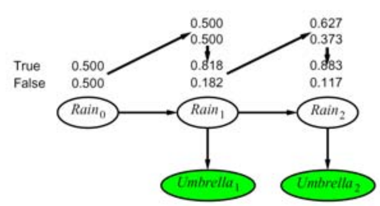
$$= \sum_{x_{t-1}} P(x_{t-1}, x_t, e_{1:t})$$

$$= \sum_{x_{t-1}} P(x_{t-1}, e_{1:t-1}) P(x_t|x_{t-1}) P(e_t|x_t)$$

$$= P(e_t|x_t) \sum_{x_{t-1}} P(x_t|x_{t-1}) P(x_{t-1}, e_{1:t-1})$$

Example

$$P(x_t|e_{1:t}) \propto f_t[x_t] = P(x_t, e_{1:t})$$



$$f_t[x_t] = P(e_t|x_t) \sum_{x_{t-1}} P(x_t|x_{t-1}) f_{t-1}[x_{t-1}]$$

Viterbi Algorithm

- Question: what is the most likely state sequence given the observations?
  - Slow answer: enumerate all possibilities
  - Better answer: cached incremental version

$$x^*_{1:T} = \arg\max_{x_{1:T}} P(x_{1:T}|e_{1:T})$$

$$\begin{aligned} m_t[x_t] &= \max_{x_{1:t-1}} P(x_{1:t-1}, x_t, e_{1:t}) \\ &= \max_{x_{1:t-1}} P(x_{1:t-1}, e_{1:t-1}) P(x_t|x_{t-1}) P(e_t|x_t) \\ &= P(e_t|x_t) \max_{x_{t-1}} P(x_t|x_{t-1}) \max_{x_{1:t-2}} P(x_{1:t-1}, e_{1:t-1}) \\ &= P(e_t|x_t) \max_{x_{t-1}} P(x_t|x_{t-1}) m_{t-1}[x_{t-1}] \end{aligned}$$

Example