Game theory: study of strategic situations, usually simultaneous actions

A game has:
- Players
- Actions
- Payoff matrix

Example: prisoner’s dilemma

Prisoner’s Dilemma

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Strategies

- **Strategy** = policy

- **Pure strategy**
  - Deterministic policy
  - In a one-move game, just a move

- **Mixed strategy**
  - Randomized policy
  - Ever good to use one?

- **Strategy profile**: a spec of one strategy per player

- **Outcome**: each strategy profile results in an (expected) number for each player

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Two-Finger Morra

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Dominance and Optimality

- **Strategy Dominance**:
  - A strategy $s$ for A (strictly) dominates $s'$ if it produces a better outcome for A, for any B strategy

- **Outcome Dominance**:
  - An outcome $o$ Pareto dominates $o'$ if all players prefer $o$ to $o'$
  - An outcome is Pareto optimal if there is no outcome that all players would prefer

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Equilibria

- In the prisoner’s dilemma:
  - What will A do?
  - What will B do?
  - What’s the dilemma?

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- Both testifying is a (Nash) equilibrium
  - Neither player can benefit from a unilateral change in strategy
  - I.e., it’s a local optimum (not necessarily global)
  - Nash showed that every game has such an equilibrium
  - Note: not every game has a dominant strategy equilibrium

- What do we have to change for the prisoners to refuse?
  - Change the payoffs
  - Consider repeated games
  - Limit the computational ability of the agents
  - How would we model a “code of thieves”?

Coordination Games

- No dominant strategy
  - But, two (pure) Nash equilibria

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<tr>
<td>DVD</td>
<td>5,5</td>
<td>8,8</td>
</tr>
<tr>
<td>HD-DVD</td>
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- What should agents do?
  - Can sometimes choose Pareto optimal Nash equilibrium
  - But may be ties!
  - Naturally gives rise to communication
  - Also: correlated equilibria

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Mixed Strategy Games

- What's the Nash equilibrium?
  - No pure strategy equilibrium
  - Must look at mixed strategies

- Mixed strategies:
  - Distribution over actions per state
  - In a one-move game, a single distribution
  - For Morra, a single number \( p_{\text{even}} \) specifies the strategy

- How to choose the optimal mixed strategy?

(Zero-Sum) Minimax Strategies

- Idea: force one player to choose and declare a strategy first
  - Say E reveals first
  - For each E strategy, O has a minimax response
  - Utility of the root favors O (why?) and is -3 (from E's perspective)
  - If O goes first, root is 2 (for E)
  - If these two utilities matched, we would know the utility of the maximum equilibrium

- Must look at mixed strategies
Continuous Minimax

- Imagine a minimax tree:
  - Instead of the two pure strategies, first player has infinitely many mixed ones
  - Note that second player should always respond with a pure strategy (why?)

- Here, can calculate the minimax (and maximin) values
  - Both are $\frac{1}{2}$ (from O’s perspective)
  - Correspond to $[7/12; 1, 5/12; 2]$ for both players

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Repeated Games

- What about repeated games?
  - E.g. repeated prisoner’s dilemma
  - Future responses, retaliation becomes an issue
  - Strategy can condition on past experience

- Repeated prisoner’s dilemma
  - Fixed numbers of games causes repeated betrayal
  - If agents unsure of number of future games, other options
    - E.g. perpetual punishment: silent until you’re betrayed, then testify thereafter
    - E.g. tit-for-tat: do what was done to you last round
  - It’s enough for your opponent to believe you are incapable of remembering the number of games played (doesn’t actually matter whether the limitation really exists)
Partially Observed Games

- Much harder to analyze
  - You have to work with trees of belief states
  - Problem: you don’t know your opponent’s belief state!

- Newer techniques can solve some partially observable games
  - Mini-poker analysis shows, e.g., that bluffing can be a rational action
  - Randomization: not just for being unpredictable, also useful for minimizing what opponent can learn from your actions

The Ultimatum Game

- Game theory can reveal interesting issues in social psychology

- E.g. the ultimatum game
  - Proposer: receives $x, offers split $k / $(x-k)
  - Acceptor: either
    - Accepts: gets $k, proposer gets $(x-k)
    - Rejects: neither gets anything

- Nash equilibrium?
  - Any strategy profile where proposer offers $k and accepter will accept $k or greater
  - But that’s not the interesting part…

- Issues:
  - Why do people tend to reject offers which are very unfair (e.g. $20 from $100)?
  - Irrationality?
  - Utility of $20 exceeded by utility of punishing the unfair proposer?
  - What about if x is very very large?
Mechanism Design

- **One use of game theory: mechanism design**
  - Designing a game which induces desired behavior in rational agents

- **E.g. avoiding tragedies of the commons**
  - Classic example: farmers share a common pasture
  - Each chooses how many goats to graze
  - Adding a goat gains utility for that farmer
  - Adding a goat slightly degrades the pasture
  - Inevitable that each farmer will keep adding goats until the commons is destroyed (tragedy!)

- **Classic solution: charge for use of the commons**
  - Prices need to be set to produce the right behavior

Auctions

- **Example: auctions**
  - Consider auction for one item
  - Each bidder $i$ has value $v_i$ and bids $b_i$ for item

- **English auction: increasing bids**
  - How should bidder $i$ bid?
  - What will the winner pay?
  - Why is this not an optimal result?

- **Sealed single-bid auction, highest pays bid**
  - How should bidder $i$ bid?
  - Why is bidding your value no longer dominant?
  - Why is this auction not optimal?

- **Sealed single-bid second-price auction**
  - How should bidder $i$ bid?
  - Bid $v_i$ – why?