CS188: Artificial Intelligence
Self-Diagnostic: Math

These questions check your level of mathematical preparedness for this course. Most students will be able to answer all of these questions. You can check your programming preparedness by doing Project 1: Search. *You do not need to hand in this self-diagnostic.*

Probabilistic inference: Your box of cereal may be a contest winner! It's rattling, which 100% of winning boxes do. Of course 1% of all boxes rattle and only one box in a million is a winner. What is the probability that your box is a winner?

According to Bayes' rule:

$P(\text{win}|\text{rattle}) = \frac{P(\text{rattle}|\text{win}) \cdot P(\text{win})}{P(\text{rattle})}$, $P(\text{win}) = \frac{1}{1000000}$, $P(\text{rattle}|\text{win}) = 1$, and $P(\text{rattle}) = \frac{10000}{1000000}$
$P(\text{win}|\text{rattle}) = 1 \cdot \frac{\frac{1}{1000000}}{\frac{10000}{1000000}} = \frac{1}{10000}$

Events: You are playing a solitaire game in which you are dealt three cards without replacement from a simplified deck of 10 (marked 1 through 10). You win if all your cards are odd or if one of them is a 10. How many winning hands are there if different orders are different hands? What is your chance of winning?

We count the size of various event types:

\# hands that are all odd $= 5 \cdot 4 \cdot 3 = 60$
\# hands that contain 10 $= 1 \cdot 9 \cdot 8 + 9 \cdot 1 \cdot 8 + 9 \cdot 8 \cdot 1 = 3 \cdot 9 \cdot 8 = 216$
\# hands that are all odd and contain 10 $= 0$
\# winning hands $= 276$
\# hands $= 10 \cdot 9 \cdot 8 = 720$
$P(\text{win}) = \frac{276}{720}$

Expectations: Someone rolls a fair six-sided die and you win points equal to the number shown. What is the expected number of points after one roll? After 2 rolls? After 100 rolls?

Let X = outcome of a single roll:

expected points after 1 roll $= E[X] = 1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} + ... + 6 \cdot \frac{1}{6} = 3.5$
eexpected points after 2 rolls $= E[X + X] = E[X] + E[X] = 7$ by linearity of expectation
expected points after n rolls $= E[nX] = nE[X] = (3.5)n$

Conditional Probabilities: Which of the following statements are true for all joint distributions over X and Y: (a) $P(x, y) = P(x)P(y)$, (b) $P(x, y) = P(x|y)P(y)$, (c) $P(x, y) = P(x|y)P(y|x)$, (d) $P(x) = \sum_y P(x|y)$, (e) $P(x) = \sum_y P(x, y)$?

Answers from definitions of independence, conditional probability, and marginal probability:

(a) FALSE: the definition of independence of X and Y, so false in general
(b) TRUE: the definition of conditional probability / the product rule
(c) FALSE: compare to (b)
(d) FALSE: try $P(x|y) = 1$ for all y

(e) TRUE: the definition of marginal probability

Linear Equations: You know that $x = (1/2)y + (1/2)(x + 1)$ and $y = (1/3)y + (1/3)(x + 2)$. What are x and y?

Many ways to solve: e.g. isolate $x = y + 1$ and substitute. Answer: $x = 4$, $y = 3$

Logarithms: Which of the following statements are true: (a) $2^{(x*y)} = 2^x 2^y$, (b) $2^{(x+y)} = 2^x 2^y$, (c) $2^{(x+y)} = 2^x + 2^y$, (d) $\log(3^x) = \log(3)\log(x)$, (e) $\log(3^x) = x\log(3)$, (f) $\log(3^x) = 3x$?

From definitions and properties of logarithms: only (b) and (e) are true.
(a) FALSE: equals $(2^x)^y$

Hashing: What critical operation is generally faster in a hashtable than in a linked list, and how fast is it typically in each? When will a hashtable degrade to the speed of a list?

Testing membership in a hashtable (contains) takes expected O(1) compared to O(n) in a linked list. The performance of contains in the hash table will degrade to O(n) if the hash function sends O(n) elements to the same hash value. This can happen if the hashing function is poorly chosen (e.g. all keys hash to the same value), or if the hashtable has too few buckets.

Induction: Prove by induction that the sum of the first n odd integers is n^2.

By induction on n:

Let S_n represent the sum of the first n odd integers, note that nth odd integer is $2n - 1$.

Base case: for $n = 1$, $S_1 = 1 = 1^2$

Inductive step: assume $S_{(n-1)} = (n-1)^2$
then $S_n = S_{(n-1)} + (2n-1) = (n-1)^2 + (2n-1) = n^2 - 2n + 1 + 2n - 1 = n^2$

By induction $S_n = n^2$ for all n