CS 188: Artificial Intelligence

Markov Decision Processes

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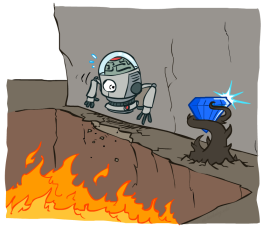
Non-Deterministic Search

http://bit.ly/188lec10mdp



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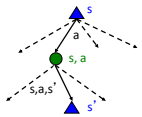
Recap: Defining MDPs

- Markov decision processes:
  - Set of states S
  - Start state s_0
  - Set of actions A
  - Transitions P(s' | s,a) (or T(s,a,s'))
  - Rewards R(s,a,s') (and discount γ)



- MDP quantities so far:
  - Policy = Choice of action for each state
  - Utility = sum of (discounted) rewards

10				1
a	b	c	d	e

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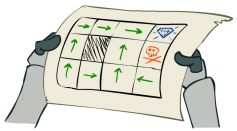
Policies

- In deterministic single-agent search problems, we wanted an optimal plan, or sequence of actions, from start to a goal
- For MDPs, we want an optimal policy $\pi^*: S \rightarrow A$
  - A policy π gives an action for each state
  - An optimal policy is one that maximizes expected utility if followed
  - An explicit policy defines a reflex agent



Optimal policy when R(s, a, s') = -0.03 for all non-terminals s

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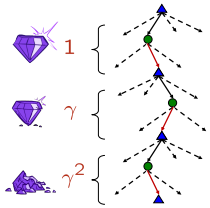
Discounting

- How to discount?
  - Each time we descend a level, we multiply in the discount once
- Why discount?
  - Think of it as a gamma chance of ending the process at every step
  - Also helps our algorithms converge
- Example: discount of 0.5
  - U([1,2,3]) = 1*1 + 0.5*2 + 0.25*3
  - U([1,2,3]) < U([3,2,1])



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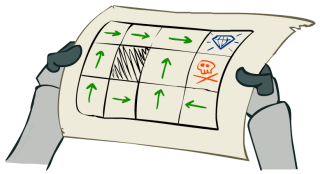
Solving MDPs



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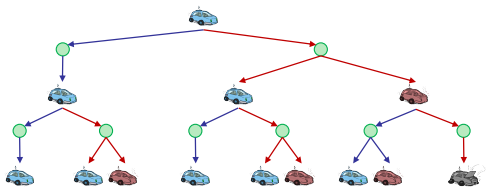
Racing Search Tree



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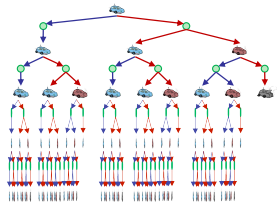
Racing Search Tree



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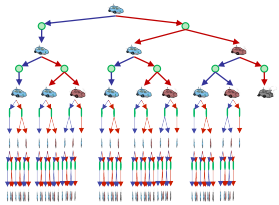
Racing Search Tree

- We're doing way too much work with expectimax!
- Problem: States are repeated
  - Idea: Only compute needed quantities once
- Problem: Tree goes on forever
  - Idea: Do a depth-limited computation, but with increasing depths until change is small
  - Note: deep parts of the tree eventually don't matter if $\gamma < 1$



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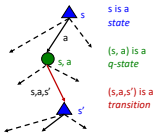
Optimal Quantities

- The value (utility) of a state s:
  $V^*(s)$ = expected utility starting in s and acting optimally
- The value (utility) of a q-state (s,a):
  $Q^*(s,a)$ = expected utility starting out having taken action a from state s and (thereafter) acting optimally
- The optimal policy:
  $\pi^*(s)$ = optimal action from state s



[Demo – gridworld values (L8D4)]

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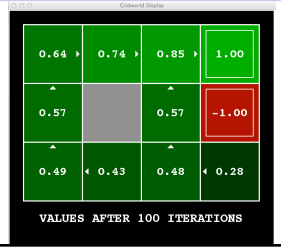
Snapshot of Demo – Gridworld V Values



Noise = 0.2
Discount = 0.9
Living reward = 0

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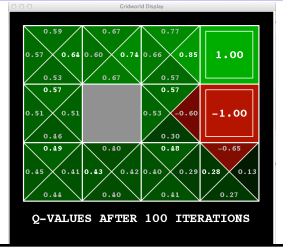
Snapshot of Demo – Gridworld Q Values



Noise = 0.2
Discount = 0.9
Living reward = 0

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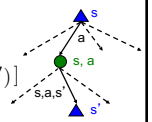
Values of States

- Recursive definition of value:

$$V^*(s) = \max_a Q^*(s,a)$$

$$Q^*(s,a) = \sum_{s'} T(s,a,s')[R(s,a,s') + \gamma V^*(s')]$$



$$V^*(s) = \max_a \sum_{s'} T(s,a,s')[R(s,a,s') + \gamma V^*(s')]$$

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Bellman Equations

- Recursive definition of value:

$$V^*(s) = \max_a \sum_{s'} T(s,a,s')[R(s,a,s') + \gamma V^*(s')]$$

- Bellman Equation:

Necessary condition for optimality in optimization problems formulated as ***Dynamic Programming***

Richard E. Bellman (1920–1984)

- Dynamic Programing:

Process to simplify an optimization problem by breaking it down into an optimal substructure.

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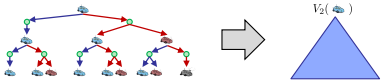
Time-Limited Values

- Key idea: time-limited values
- Define $V_k(s)$ to be the optimal value of s if the game ends in k more time steps
  - Equivalently, it's what a depth-k expectimax would give from s



[Demo – time-limited values (L8D6)]

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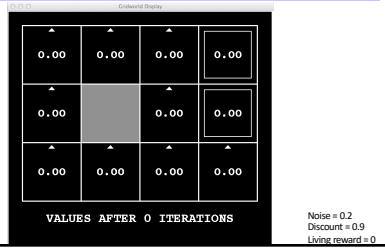
k=0



Noise = 0.2
Discount = 0.9
Living reward = 0

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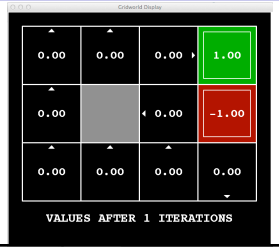
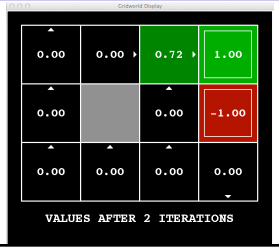
k=1



Noise = 0.2
Discount = 0.9
Living reward = 0

17

k=2



Noise = 0.2
Discount = 0.9
Living reward = 0

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k=3

19

k=4

20

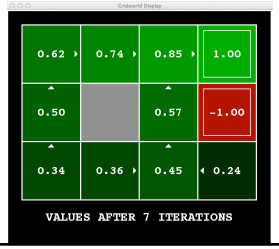
k=5

21

k=6

22

k=7



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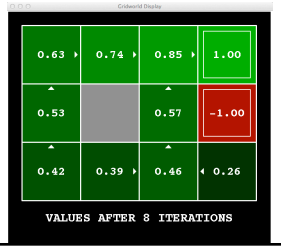
k=8



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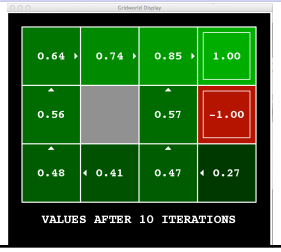
k=9

Noise = 0.2
Discount = 0.9
Living reward = 0

25

k=10



Noise = 0.2
Discount = 0.9
Living reward = 0

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k=11

Noise = 0.2
Discount = 0.9
Living reward = 0

27

k=12

Noise = 0.2
Discount = 0.9
Living reward = 0

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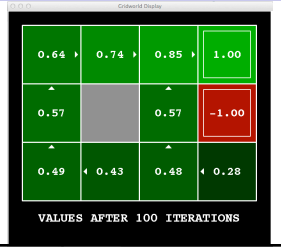
k=100



Noise = 0.2
Discount = 0.9
Living reward = 0

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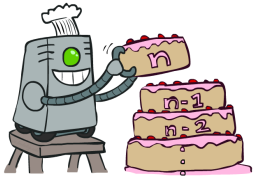
Value Iteration



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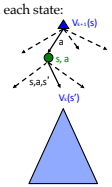
Value Iteration

- Start with $V_0(s) = 0$: no time steps left means an expected reward sum of zero
- Given vector of $V_k(s)$ values, do one ply of expectimax from each state:

$$V_{k+1}(s) \leftarrow \max_a \sum_{s'} T(s,a,s') \left[R(s,a,s') + \gamma V_k(s')\right]$$



- Repeat until convergence
- Complexity of each iteration: $O(S^2A)$
- Theorem: will converge to unique optimal values
  - Basic idea: approximations get refined towards optimal values
  - Policy may converge long before values do

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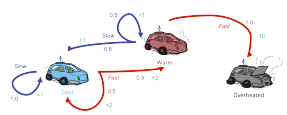
Example: Value Iteration

V_2			
V_1	S: 1 F: .5*2+.5*2=2		
V_0	0	0	0



Assume no discount!

$$V_{k+1}(s) \leftarrow \max_a \sum_{s'} T(s,a,s') \left[R(s,a,s') + \gamma V_k(s')\right]$$

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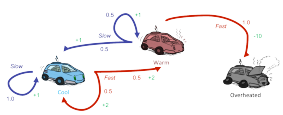
Example: Value Iteration

V_2			
V_1	2	S: .5*1+.5*1=1 F: -10	
V_0	0	0	0



Assume no discount!

$$V_{k+1}(s) \leftarrow \max_a \sum_{s'} T(s,a,s') \left[R(s,a,s') + \gamma V_k(s')\right]$$

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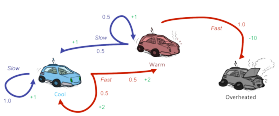
Example: Value Iteration

V_2			
V_1	2	1	0
V_0	0	0	0



Assume no discount!

$$V_{k+1}(s) \leftarrow \max_a \sum_{s'} T(s,a,s') \left[R(s,a,s') + \gamma V_k(s')\right]$$

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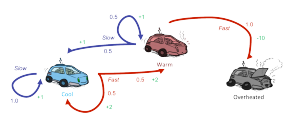
Example: Value Iteration

V_2	S: 1+2=3 F: ~~.5*(2+2)+.5*(2+1)=3.5~~		
V_1	2	1	0
V_0	0	0	0



Assume no discount!

$$V_{k+1}(s) \leftarrow \max_a \sum_{s'} T(s,a,s') \left[R(s,a,s') + \gamma V_k(s')\right]$$

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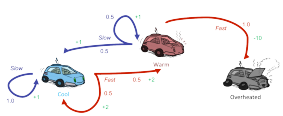
Example: Value Iteration

V_2	3.5	2.5	0
V_1	2	1	0
V_0	0	0	0



Assume no discount!

$$V_{k+1}(s) \leftarrow \max_a \sum_{s'} T(s,a,s') \left[R(s,a,s') + \gamma V_k(s')\right]$$

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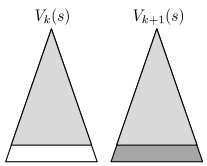
Convergence*

- How do we know the V_k vectors are going to converge?
- Case 1: If the tree has maximum depth M, then V_M holds the actual untruncated values
- Case 2: If the discount is less than 1
  - Sketch: For any state V_k and V_{k+1} can be viewed as depth k+1 expectimax results in nearly identical search trees
  - The difference is that on the bottom layer, V_{k+1} has actual rewards while V_k has zeros
  - That last layer is at best all R_{MAX}
  - It is at worst R_{MIN}
  - But everything is discounted by γ^k that far out
  - So V_k and V_{k+1} are at most γ^k max|R| different
  - So as k increases, the values converge



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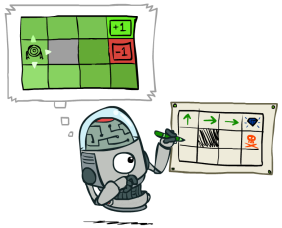
Policy Extraction



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Computing Actions from Values

- Let's imagine we have the optimal values V*(s)
- How should we act?
  - It's not obvious!
- We need to do a mini-expectimax (one step)

$$\pi^*(s) = \arg\max_a \sum_{s'} T(s,a,s')[R(s,a,s') + \gamma V^*(s')]$$

- This is called policy extraction, since it gets the policy implied by the values

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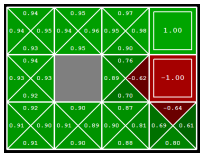
Computing Actions from Q-Values

- Let's imagine we have the optimal q-values:
- How should we act?
  - Completely trivial to decide!

$$\pi^*(s) = \arg\max_a Q^*(s,a)$$



- Important lesson: actions are easier to select from q-values than values!

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Let's think.

- Take a minute, think about value iteration.
- Write down the biggest question you have about it.

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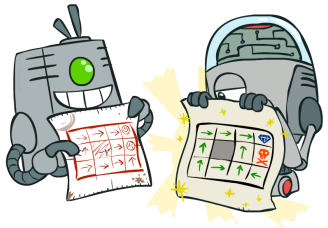
Policy Methods



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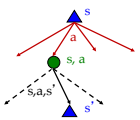
Problems with Value Iteration

- Value iteration repeats the Bellman updates:

$$V_{k+1}(s) \leftarrow \max_a \sum_{s'} T(s,a,s') \left[R(s,a,s') + \gamma V_k(s')\right]$$



- Problem 1: It's slow – $O(S^2A)$ per iteration
- Problem 2: The "max" at each state rarely changes
- Problem 3: The policy often converges long before the values

[Demo: value iteration (L9D2)]

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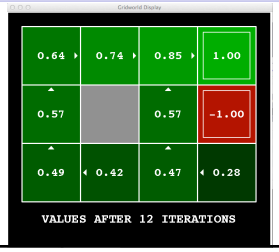
k=12



Noise = 0.2
Discount = 0.9
Living reward = 0

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k=100

Noise = 0.2
Discount = 0.9
Living reward = 0

45

Policy Iteration

- Alternative approach for optimal values:
  - Step 1: Policy evaluation: calculate utilities for some fixed policy (not optimal utilities!) until convergence
  - Step 2: Policy improvement: update policy using one-step look-ahead with resulting converged (but not optimal!) utilities as future values
  - Repeat steps until policy converges
- This is policy iteration
  - It's still optimal!
  - Can converge (much) faster under some conditions

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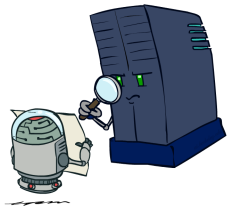
Policy Evaluation



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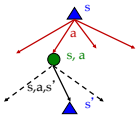
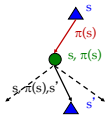
Fixed Policies

Do the optimal action



Do what π says to do



- Expectimax trees max over all actions to compute the optimal values
- If we fixed some policy $\pi(s)$, then the tree would be simpler – only one action per state
  - … though the tree's value would depend on which policy we fixed

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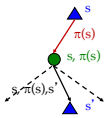
Utilities for a Fixed Policy

- Another basic operation: compute the utility of a state s under a fixed (generally non-optimal) policy
- Define the utility of a state s, under a fixed policy π:
  $V^\pi(s)$ = expected total discounted rewards starting in s and following π
- Recursive relation (one-step look-ahead / Bellman equation):

$$V^\pi(s) = \sum_{s'} T(s, \pi(s), s')[R(s, \pi(s), s') + \gamma V^\pi(s')]$$



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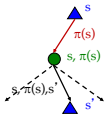
Policy Evaluation

- How do we calculate the V's for a fixed policy π?
- Idea 1: Turn recursive Bellman equations into updates (like value iteration)

$$V_0^\pi(s) = 0$$

$$V_{k+1}^\pi(s) \leftarrow \sum_{s'} T(s, \pi(s), s')[R(s, \pi(s), s') + \gamma V_k^\pi(s')]$$



- Efficiency: $O(S^2)$ per iteration
- Idea 2: Without the maxes, the Bellman equations are just a linear system
  - Solve with Matlab (or your favorite linear system solver)

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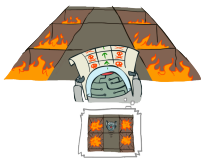
Example: Policy Evaluation

Always Go Right

Always Go Forward



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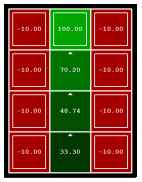
Example: Policy Evaluation

Always Go Right

Always Go Forward



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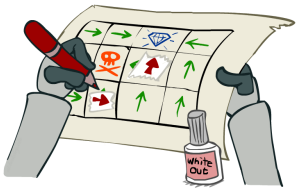
Policy Iteration



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Policy Iteration

- Evaluation: For fixed current policy π, find values with policy evaluation:
  - Iterate until values converge:

$$V_{k+1}^{\pi_i}(s) \leftarrow \sum_{s'} T(s, \pi_i(s), s') \left[R(s, \pi_i(s), s') + \gamma V_k^{\pi_i}(s')\right]$$

- Improvement: For fixed values, get a better policy using policy extraction
  - One-step look-ahead:

$$\pi_{i+1}(s) = \arg\max_a \sum_{s'} T(s, a, s') \left[R(s, a, s') + \gamma V^{\pi_i}(s')\right]$$

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Comparison

- Both value iteration and policy iteration compute the same thing (all optimal values)
- In value iteration:
  - Every iteration updates both the values and (implicitly) the policy
  - We don't track the policy, but taking the max over actions implicitly recomputes it
- In policy iteration:
  - We do several passes that update utilities with fixed policy (each pass is fast because we consider only one action, not all of them)
  - After the policy is evaluated, a new policy is chosen (slow like a value iteration pass)
  - The new policy will be better (or we're done)
- Both are dynamic programs for solving MDPs

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Summary: MDP Equations

- Value iteration equation:
  $$V_{k+1}(s) \leftarrow \max_a \sum_{s'} T(s,a,s') \left[R(s,a,s') + \gamma V_k(s')\right]$$
- Policy evaluation equation:
  $$V_{k+1}^{\pi_i}(s) \leftarrow \sum_{s'} T(s,\pi_i(s),s') \left[R(s,\pi_i(s),s') + \gamma V_k^{\pi_i}(s')\right]$$
- Policy iteration equation:
  $$\pi_{i+1}(s) = \arg\max_a \sum_{s'} T(s,a,s') \left[R(s,a,s') + \gamma V^{\pi_i}(s')\right]$$

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Summary: MDP Algorithms

- So you want to….
  - Compute optimal values: use value iteration or policy iteration
  - Compute values for a particular policy: use policy evaluation
  - Turn your values into a policy: use policy extraction (one-step lookahead)
- These all look the same!
  - They basically are – they are all variations of Bellman updates
  - They all use one-step lookahead expectimax fragments
  - They differ only in whether we plug in a fixed policy or max over actions

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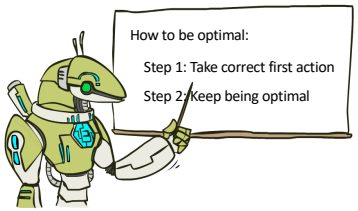
The Bellman Equations



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Next Time: Reinforcement Learning!

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