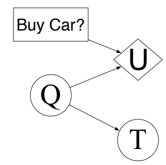
CS 188
Summer 2023

Discussion 5A Solutions

1 Decision Networks and VPI

A buyer is deciding whether to buy a certain used car. The car may be good quality ($Q = +q$) or bad quality ($Q = -q$). A test (T) costs \$50 and can help to figure out the quality of the car. There are only two outcomes for the test: $T =$ pass or $T =$ fail. The car costs \$1,500, and its market value is \$2,000 if it is good quality; if not, \$700 in repairs will be needed to make it good quality. The buyer's estimate is that the car has 70% chance of being good quality.



(a) Calculate the expected net gain from buying the car, given no test.

$$\begin{aligned} EU(\text{buy}) &= P(Q=+q)\cdot U(+q,\text{buy}) + P(Q=-q)\cdot U(-q, buy) \\ &= .7\cdot 500 + 0.3\cdot -200 = 290 \end{aligned}$$

(b) Tests can be described by the probability that the car will pass or fail the test given that the car is good or bad quality. We know: $P(T = \text{pass}|Q = +q) = 0.9$ and $P(T = \text{pass}|Q = -q) = 0.2$

Calculate the probability that the car will pass (or fail) its test, and then the probability that it is good (or bad) quality given each possible test outcome.

$$\begin{aligned} P(T=\text{pass}) &= \sum_q P(T=\text{pass}, Q=q) \\ &= P(T=\text{pass}|Q=+q)P(Q=+q) + P(T=\text{pass}|Q=-q)P(Q=-q) \\ &= 0.69 \\ P(T=\text{fail}) &= 0.31 \\ P(Q=+q|T=\text{pass}) &= \frac{P(T=\text{pass}|Q=+q)P(Q=+q)}{P(T=\text{pass})} \\ &= \frac{0.9\cdot 0.7}{0.69} = \frac{21}{23} \approx 0.91 \\ P(Q=+q|T=\text{fail}) &= \frac{P(T=\text{fail}|Q=+q)P(Q=+q)}{P(T=\text{fail})} \\ &= \frac{0.1\cdot 0.7}{0.31} = \frac{7}{31} \approx 0.22 \end{aligned}$$

(c) Calculate the optimal decisions given either a pass or a fail, and their expected utilities.

$$\begin{aligned} EU(\text{buy}|T=\text{pass}) &= P(Q=+q|T=\text{pass})U(+q,\text{buy}) + P(Q=-q|T=\text{pass})U(-q,\text{buy}) \\ &\approx 0.91\cdot 500 + 0.09\cdot(-200) \approx 437 \\ EU(\text{buy}|T=\text{fail}) &= P(Q=+q|T=\text{fail})U(+q,\text{buy}) + P(Q=-q|T=\text{fail})U(-q,\text{buy}) \\ &\approx 0.22\cdot 500 + 0.78\cdot(-200) = -46 \\ EU(\neg\text{buy}|T=\text{pass}) &= 0 \\ EU(\neg\text{buy}|T=\text{fail}) &= 0 \end{aligned}$$

Therefore: $MEU(T = \text{pass}) = 437$ (with buy) and $MEU(T = \text{fail}) = 0$ (using $\neg$buy)

(d) Calculate the value of (perfect) information of the test. Should the buyer pay for a test?

$$
\begin{aligned}
VPI(T) &= \left(\sum_t P(T=t) MEU(T=t)\right) - MEU(\varnothing) \\
&= 0.69 \cdot 437 + 0.31 \cdot 0 - 290 \approx 11.53
\end{aligned}
$$

You shouldn't pay for it, since the cost is $50.

Q2. VPI

You are the latest contestant on Monty Hall's game show, which has undergone a few changes over the years. In the game, there are n closed doors: behind one door is a car ($U(car) = 1000$), while the other $n-1$ doors each have a goat behind them ($U(goat) = 10$). You are permitted to open exactly one door and claim the prize behind it.

You begin by choosing a door uniformly at random.

(a) What is your expected utility? $(1000 * \frac{1}{n} + 10 * \frac{n-1}{n} = 10 + 990 * \frac{1}{n})$
We can calculate the expected utility via the usual formula of expectation, or we can note that there is a guaranteed utility of 10, with a small probability of a bonus utility. The latter is a bit simpler, so the answers to the following parts use this form.

(b) After you choose a door but before you open it, Monty offers to open k other doors, each of which are guaranteed to have a goat behind it. If you accept this offer, should you keep your original choice of a door, or switch to a new door? $EU(keep)$:$10 + 990 * \frac{1}{n}$

$EU(switch)$: $10 + 990 * \frac{(n-1)}{n*(n-k-1)}$

Action that achieves MEU: switch

The expected utility if we keep must be the same as the answer from the previous part: the probability that we have a winning door has not changed at all, since we have gotten no meaningful information.
In order to win a car by switching, we must have chosen a goat door previously (probability $\frac{n-1}{n}$) and then switch to the car door (probability $\frac{1}{n-k-1}$).
Since $n - 1 > n - k - 1$ for positive k, switching gets a larger expected utility.

(c) What is the value of the information that Monty is offering you? $990 * \frac{1}{n} * \frac{k}{n-k-1}$
The formula for VPI is $MEU(e) - MEU(\emptyset)$. Thus, we want the difference between $EU(switch)$ (the optimal action if Monty opens the doors) and our expected utility from part (a).
(It is true that $EU(keep)$ happens to have the same numeric expression as in part (a), but this fact is not meaningful in answering this part.)

(d) Monty is changing his offer!
After you choose your initial door, you are given the offer to choose any other door and open this second door. If you do, after you see what is inside the other door, you may switch your initial choice (to the newly opened door) or keep your initial choice.
What is the value of this new offer? $\frac{990}{n}$
Intuitively, if we take this offer, it is as if we just chose two doors in the beginning, and we win if either door has the car behind it. Unlike in the previous parts, if the new door has a goat behind it, it is not more optimal to switch doors.
Mathematically, letting D_i be the event that door i has the car, we can calculate this as $P(D_2 \cup D_1) = P(D_1) + P(D_2) = 1/n + 1/n = 2/n$, to see that $MEU(\text{offer}) = 10 + 990 * \frac{2}{n}$. Subtracting the expected utility without taking the offer, we are left with $990 * \frac{1}{n}$.

(e) Monty is generalizing his offer: you can pay $\$d^3$ to open d doors as in the previous part. (Assume that $U(\$x) = x$.) You may now switch your choice to any of the open doors (or keep your initial choice). What is the largest value of d for which it would be rational to accept the offer? $d = \sqrt{\frac{990}{n}}$
It is a key insight (whether intuitive or determined mathematically) that the answer to the previous part is constant for each successive door we open. Thus, the value of opening d doors is just $d * 990 * \frac{1}{n}$. Setting this equal to d^3, we can solve for d.