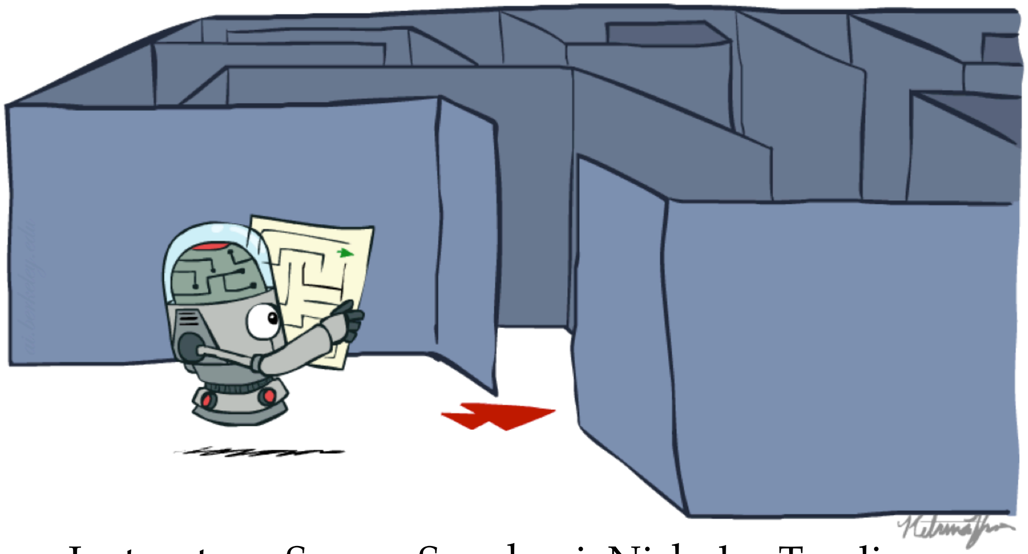
CS 188: Artificial Intelligence

Search Problems



Instructors: Saagar Sanghavi, Nicholas Tomlin

University of California, Berkeley

(slides adapted from Dan Klein, Pieter Abbeel, Anca Dragan, Stuart Russell)

Last time…

- Utilities and Rationality
- Rational Preferences
- MEU Principle

Orderability: $(A > B) \vee (B > A) \vee (A \sim B)$

Transitivity: $(A > B) \wedge (B > C) \Rightarrow (A > C)$

Continuity: $(A > B > C) \Rightarrow \exists p\ [p, A;\ 1\text{-p}, C] \sim B$

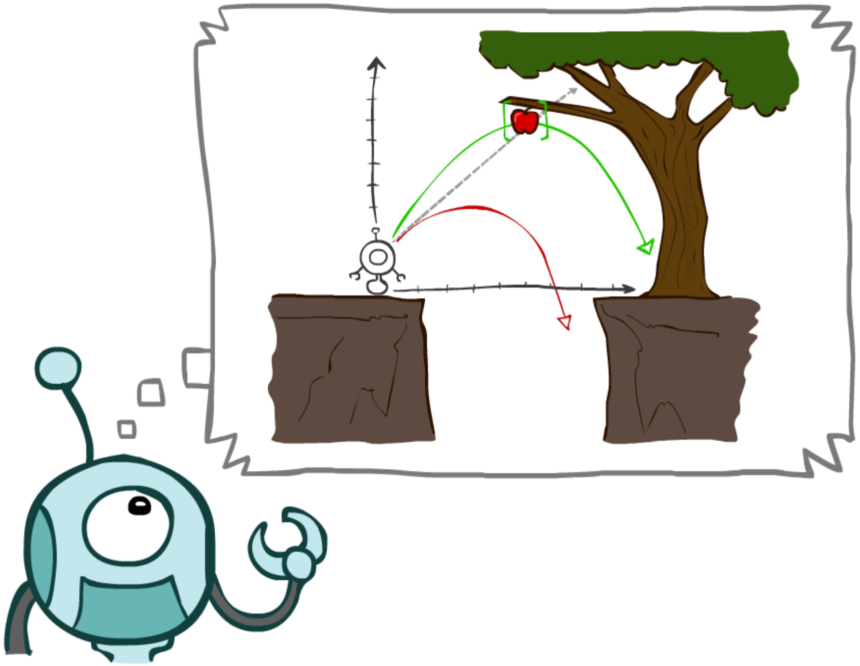
Substitutability: $(A \sim B) \Rightarrow [p, A;\ 1\text{-p}, C] \sim [p, B;\ 1\text{-p}, C]$

Monotonicity: $(A > B) \Rightarrow$

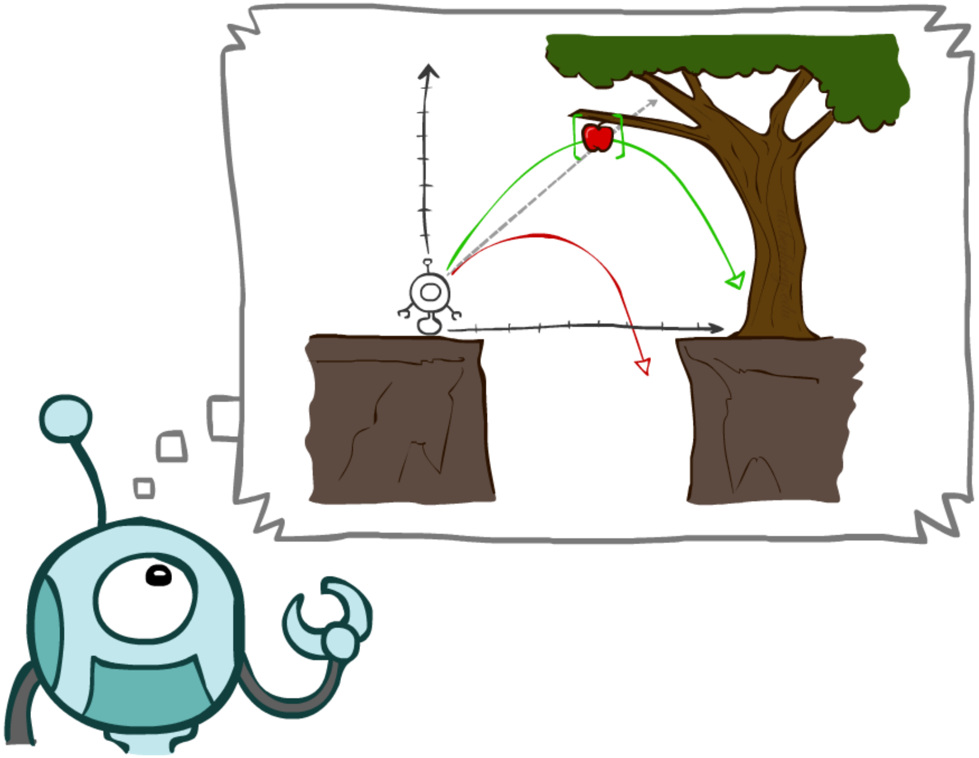
$$(p \geq q) \Leftrightarrow [p, A;\ 1\text{-p}, B] \geq [q, A;\ 1\text{-q}, B]$$

Today

- Agents that Plan Ahead
- Search Problems
- Uninformed Search Methods
  - Depth-First Search
  - Breadth-First Search
  - Uniform-Cost Search

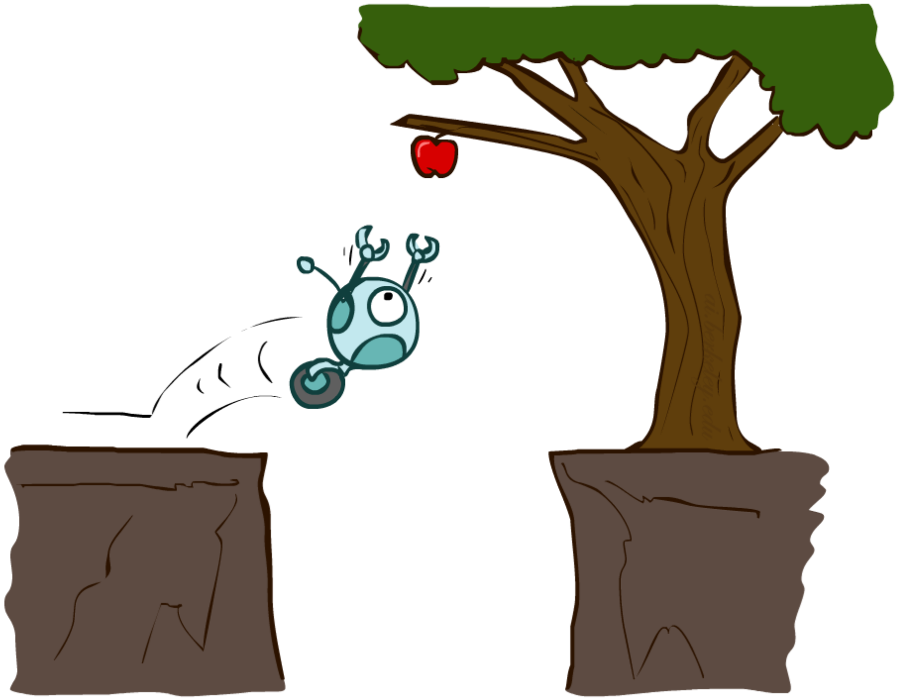


Agents that Plan

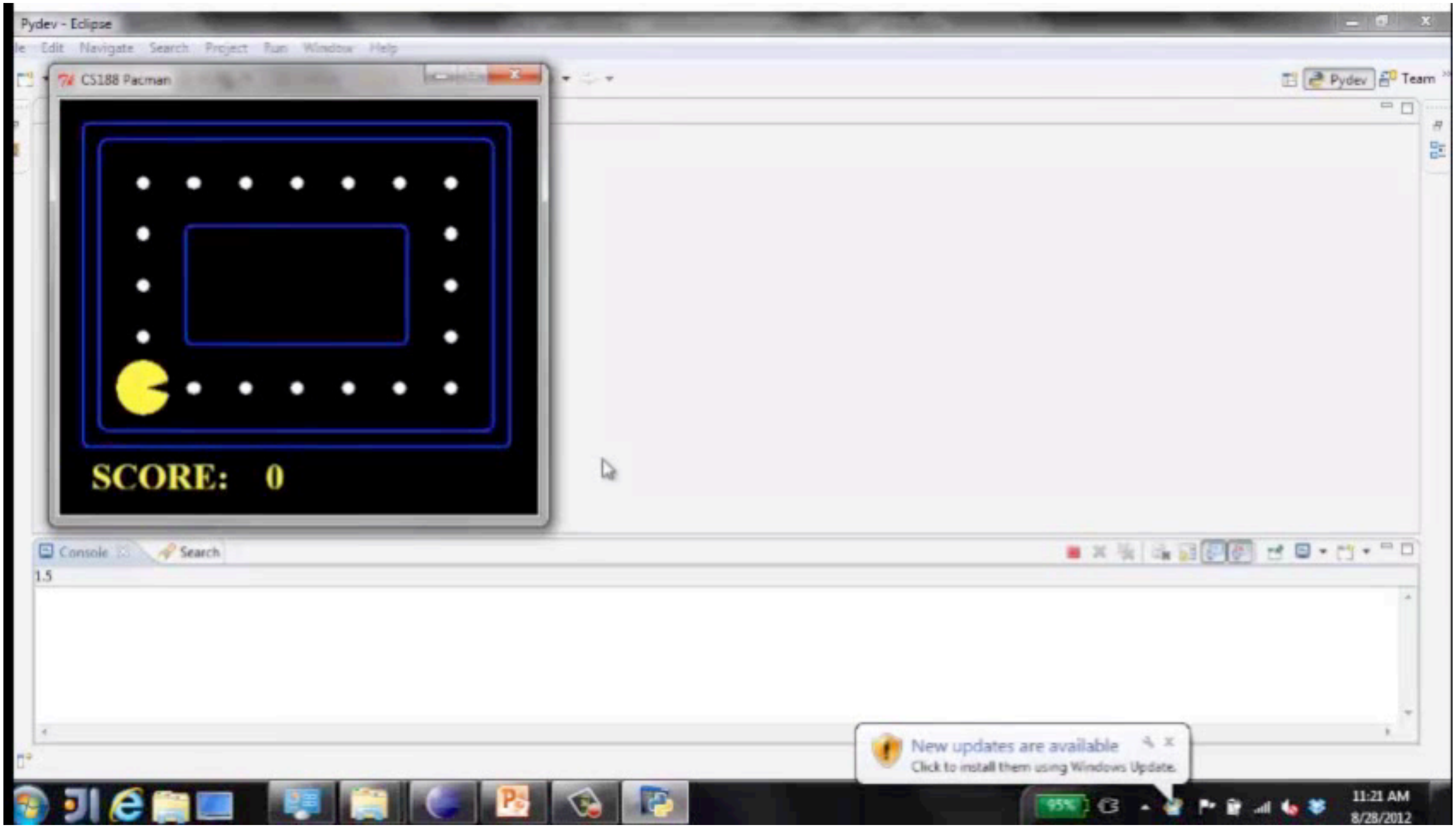


Reflex Agents

- Reflex agents:
  - Choose action based on current percept (and maybe memory)
  - May have memory or a model of the world's current state
  - Do not consider the future consequences of their actions
  - Consider how the world IS
- Can a reflex agent be rational?



Video of Demo Reflex Optimal



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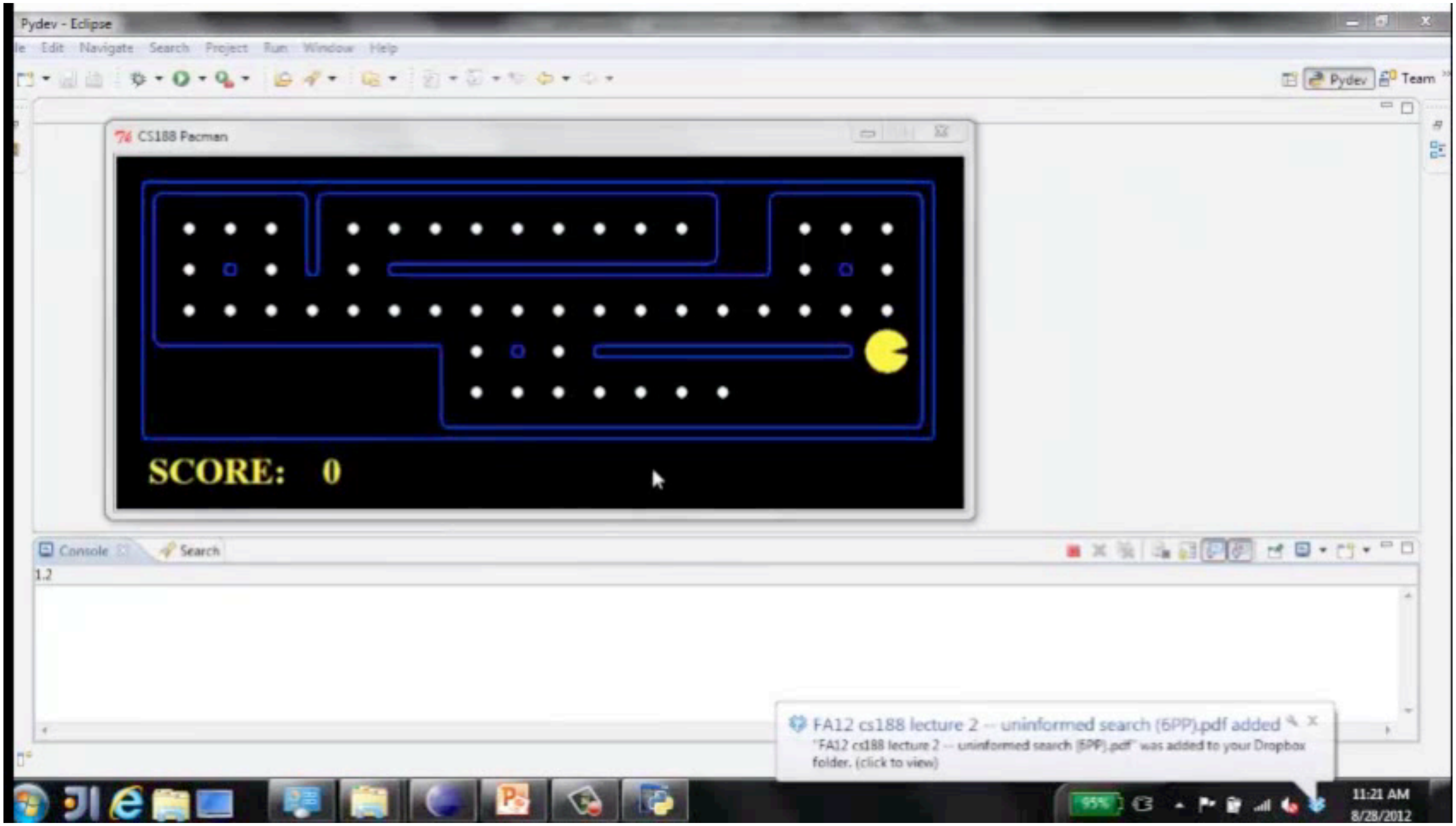
New updates are available
Click to install them using Windows Update.

Video of Demo Reflex Odd



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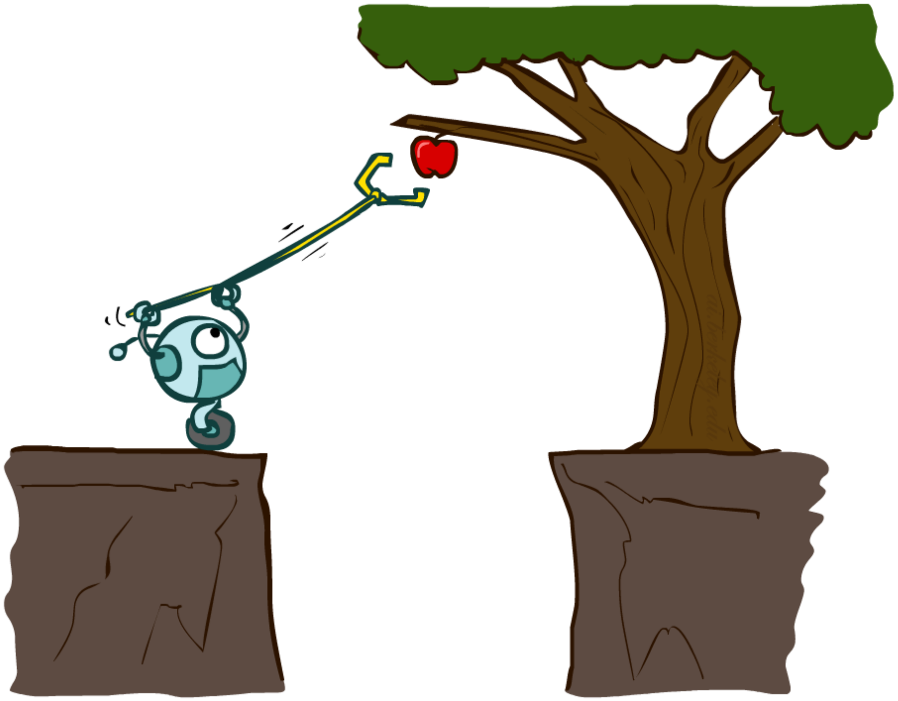
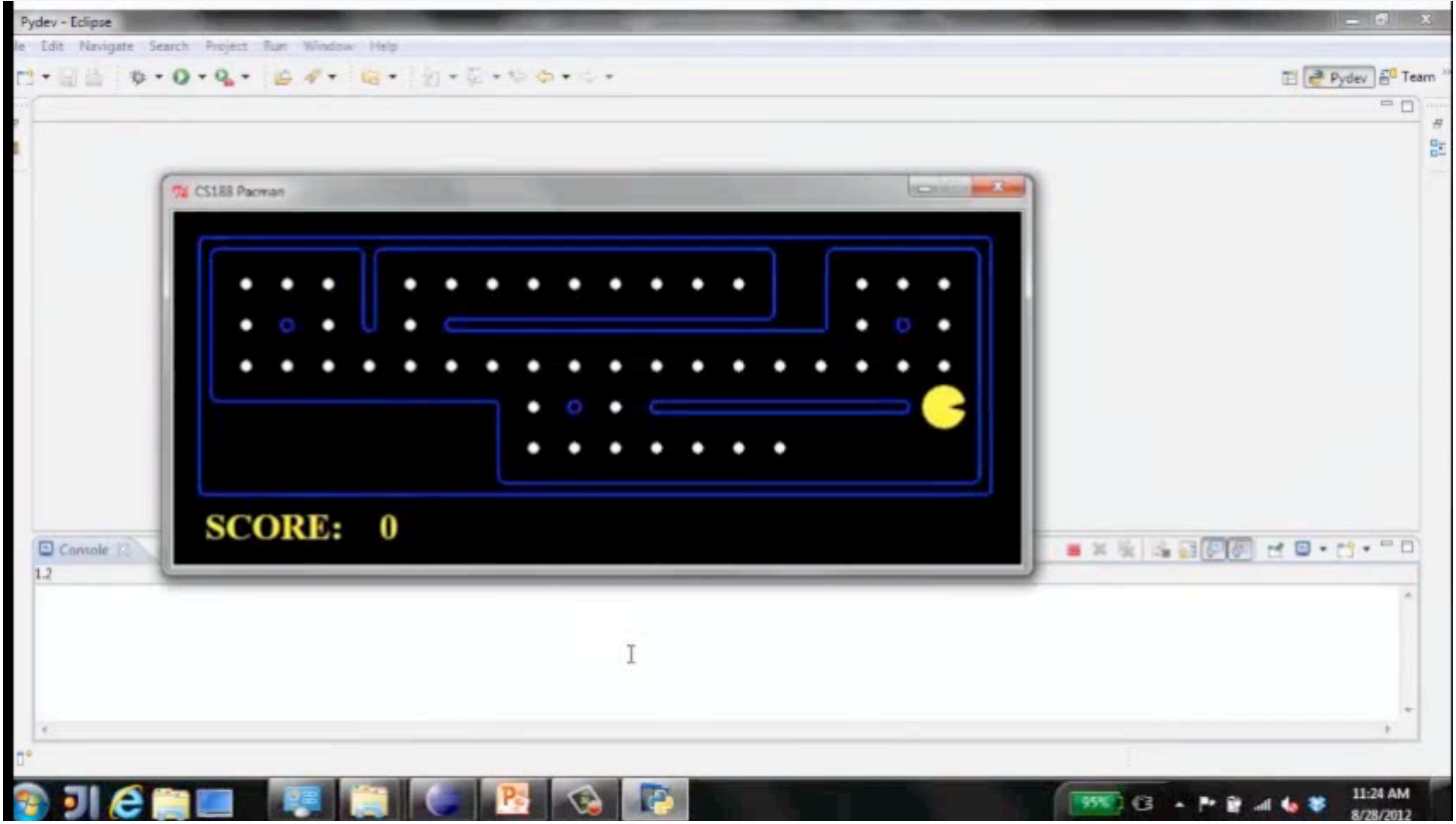
FA12 cs188 lecture 2 -- uninformed search (6PP).pdf added
"FA12 cs188 lecture 2 -- uninformed search (6PP).pdf" was added to your Dropbox folder. (click to view)

Planning Agents

- Planning agents:
    - Ask "what if"
    - Decisions based on (hypothesized) consequences of actions
    - Must have a model of how the world evolves in response to actions
    - Must formulate a goal (test)
    - Consider how the world WOULD BE
- Optimal vs. complete planning
- Planning vs. replanning



Video of Demo Replanning



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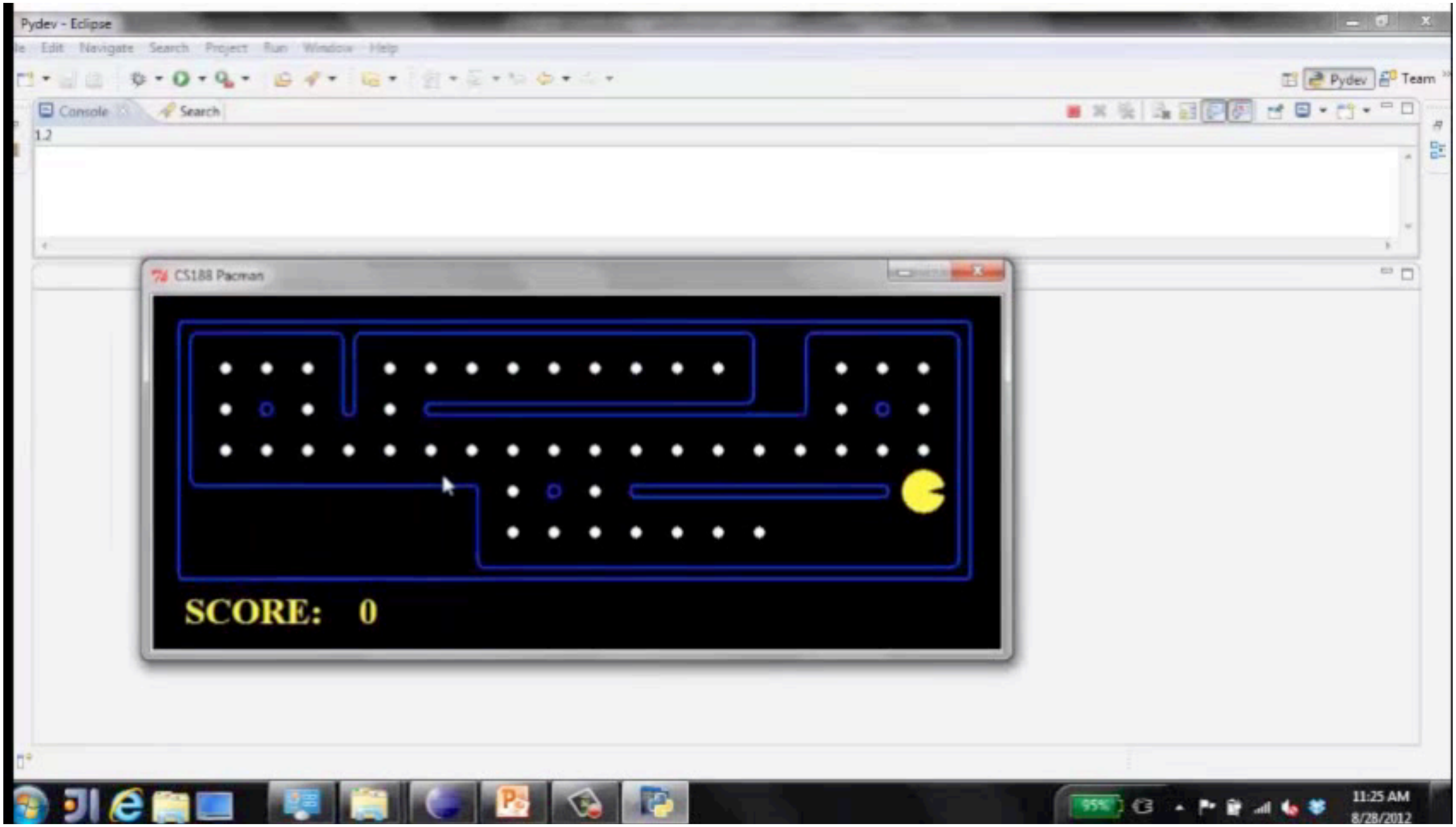
Video of Demo Mastermind



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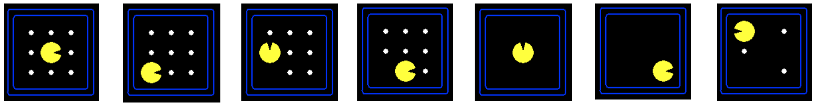
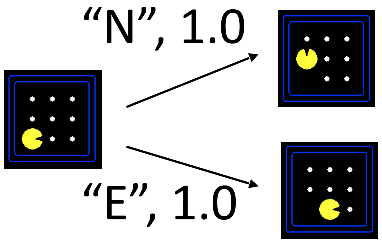
Search Problems

Search Problems

- A search problem consists of:
  - A state space



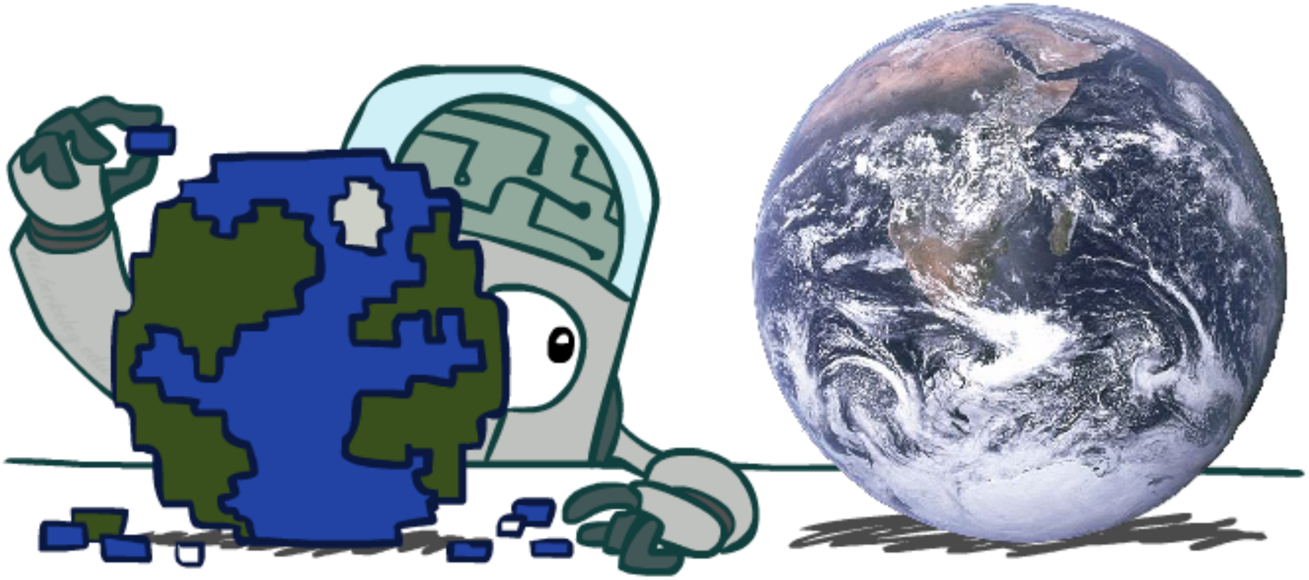
  - A successor function
    (with actions, costs)



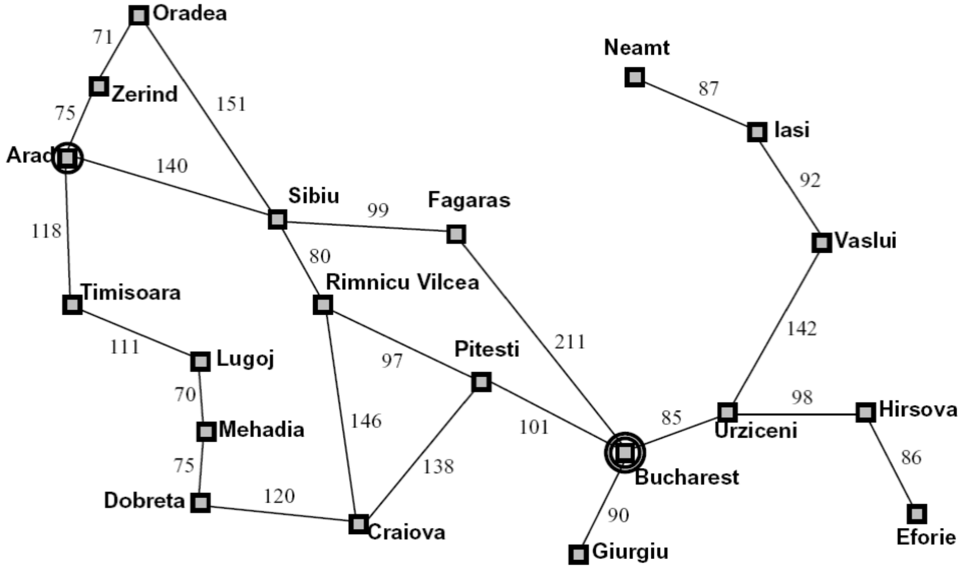
  - A start state and a goal test

- A solution is a sequence of actions (a plan) which transforms the start state to a goal state

Search Problems Are Models



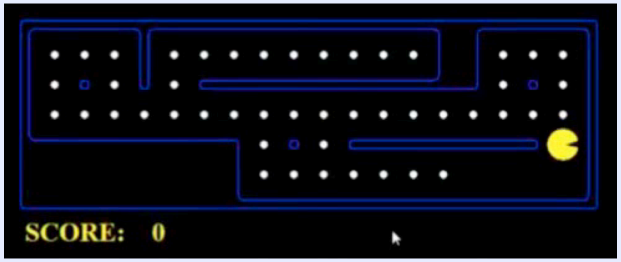
Example: Traveling in Romania



- State space:
  - Cities
- Successor function:
  - Roads: Go to adjacent city with cost = distance
- Start state:
  - Arad
- Goal test:
  - Is state == Bucharest?
- Solution?

What's in a State Space?

The world state includes every last detail of the environment



A search state keeps only the details needed for planning (abstraction)

- Problem: Pathing
  - States: (x,y) location
  - Actions: NSEW
  - Successor: update location only
  - Goal test: is (x,y)=END

- Problem: Eat-All-Dots
  - States: {(x,y), dot booleans}
  - Actions: NSEW
  - Successor: update location and possibly a dot boolean
  - Goal test: dots all false

State Space Sizes?

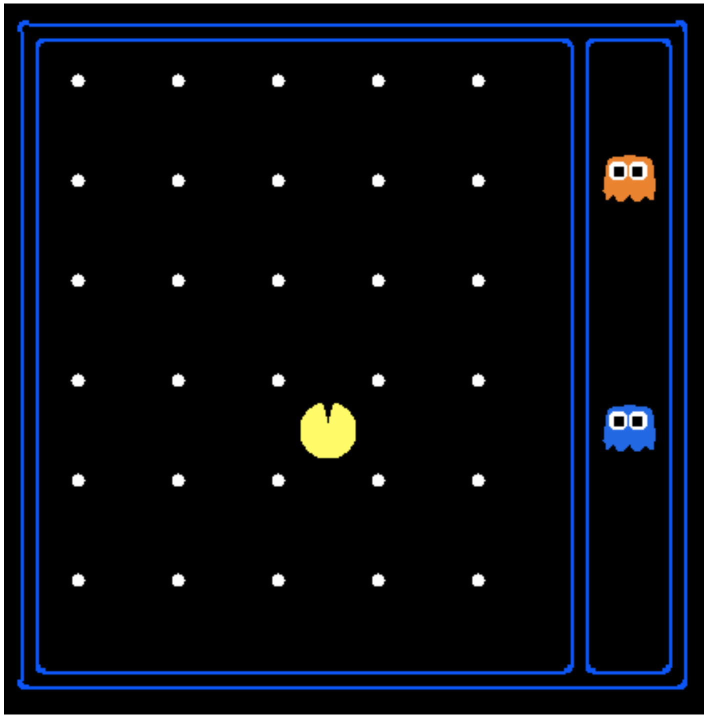
- World state:
  - Agent positions: 120
  - Food count: 30
  - Ghost positions: 12
  - Agent facing: NSEW

- How many
  - World states?

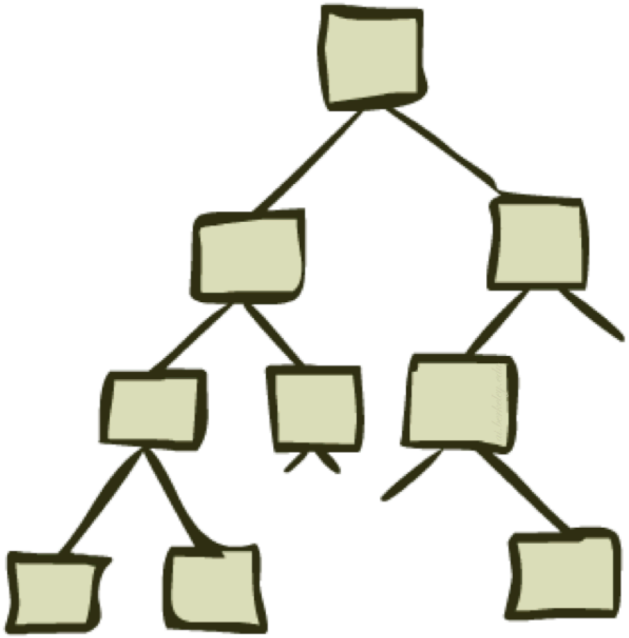
    $120x(2^{30})x(12^{2})x4$
  - States for pathing?

    120
  - States for eat-all-dots?

    $120x(2^{30})$

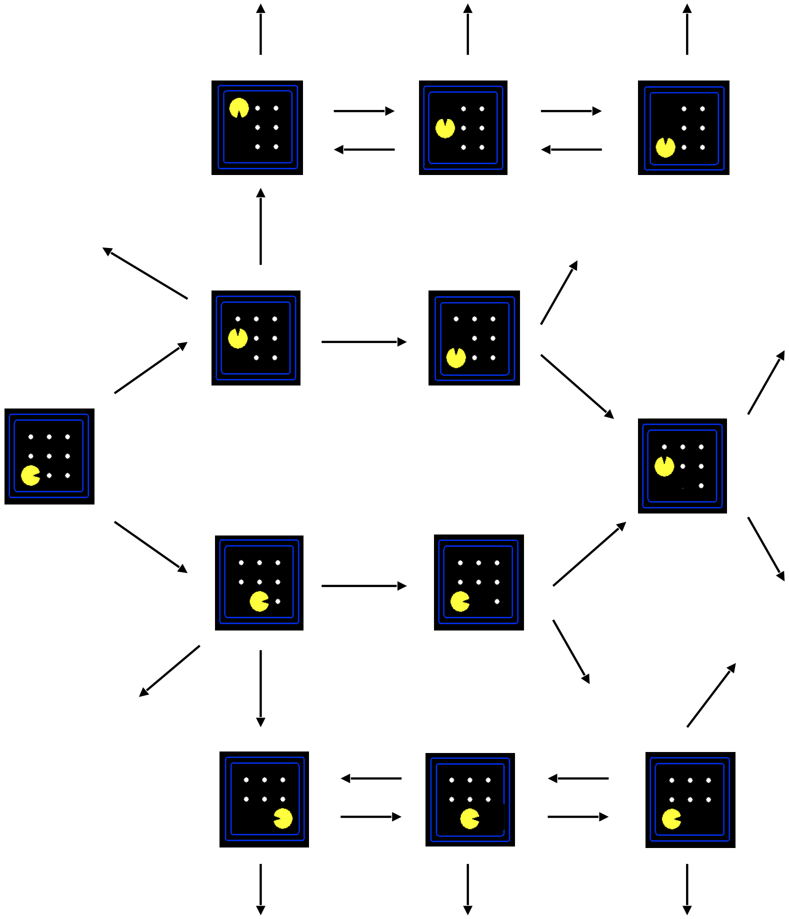


State Space Graphs and Search Trees



State Space Graphs

- State space graph: A mathematical representation of a search problem
  - Nodes are (abstracted) world configurations
  - Arcs represent successors (action results)
  - The goal test is a set of goal nodes (maybe only one)
- In a state space graph, each state occurs only once!
- We can rarely build this full graph in memory (it's too big), but it's a useful idea



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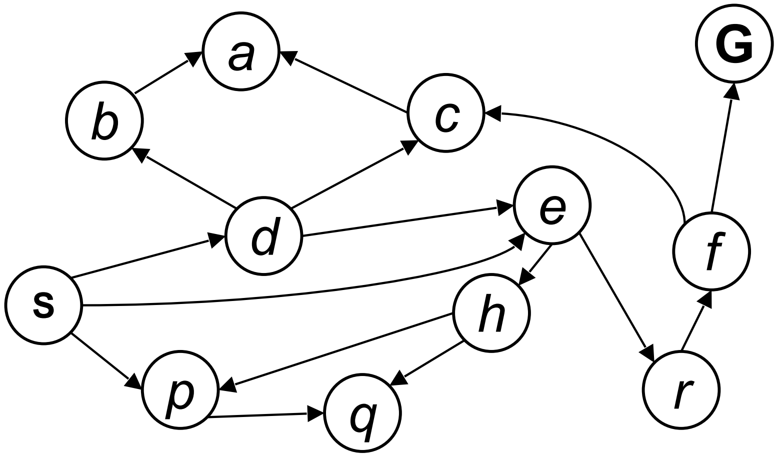
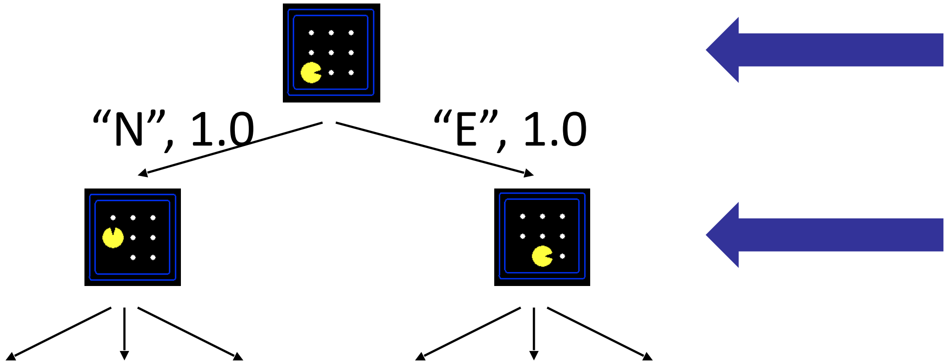


Tiny state space graph for a tiny search problem

Search Trees

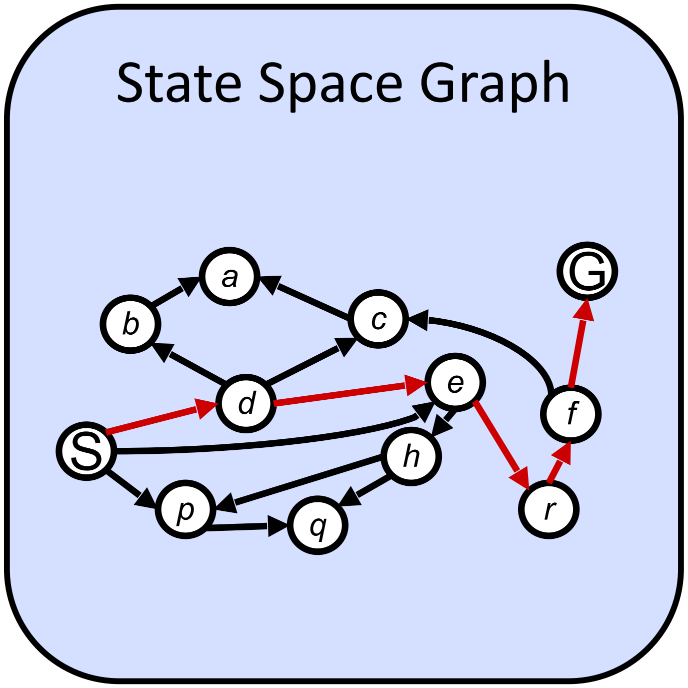


This is now / start

Possible futures

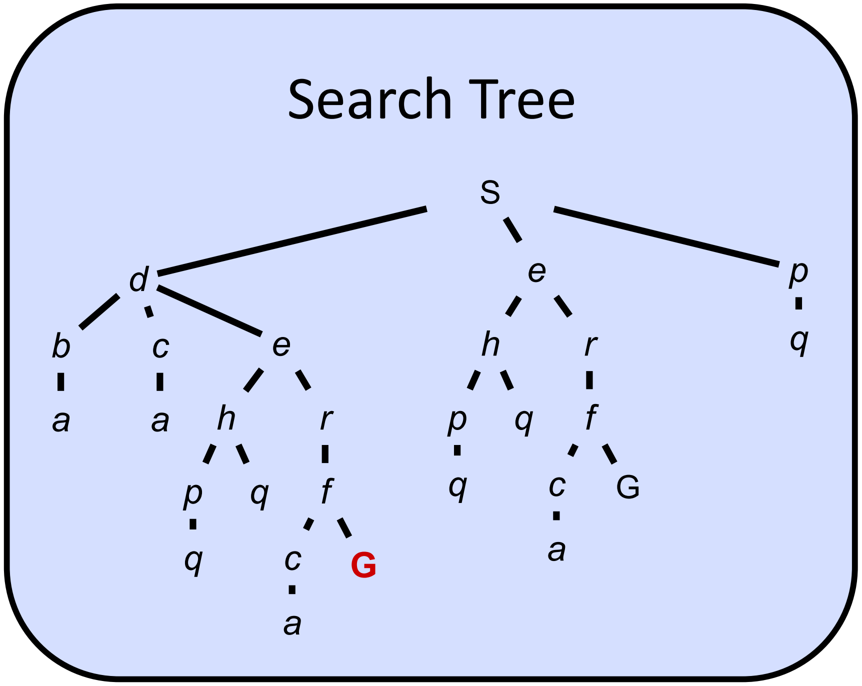
- A search tree:
  - A "what if" tree of plans and their outcomes
  - The start state is the root node
  - Children correspond to successors
  - Nodes show states, but correspond to PLANS that achieve those states
  - For most problems, we can never actually build the whole tree

State Space Graphs vs. Search Trees



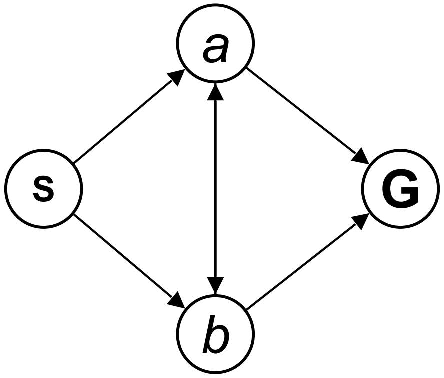
Each NODE in in the search tree is an entire PATH in the state space graph.

We construct only what we need on demand



Quiz: State Space Graphs vs. Search Trees

Consider this 4-state graph:

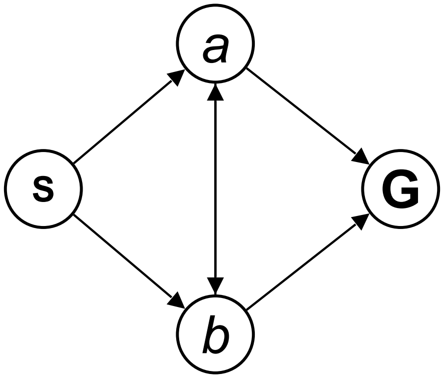


How big is its search tree (from S)?

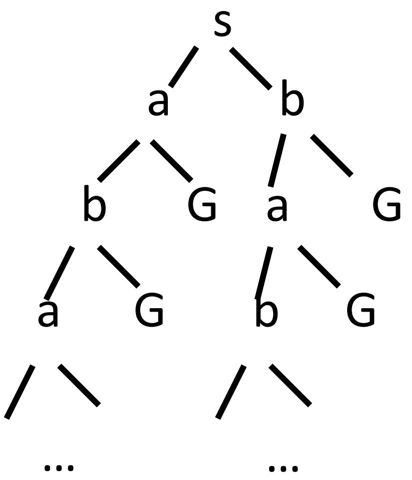
∞

Quiz: State Space Graphs vs. Search Trees

Consider this 4-state graph:



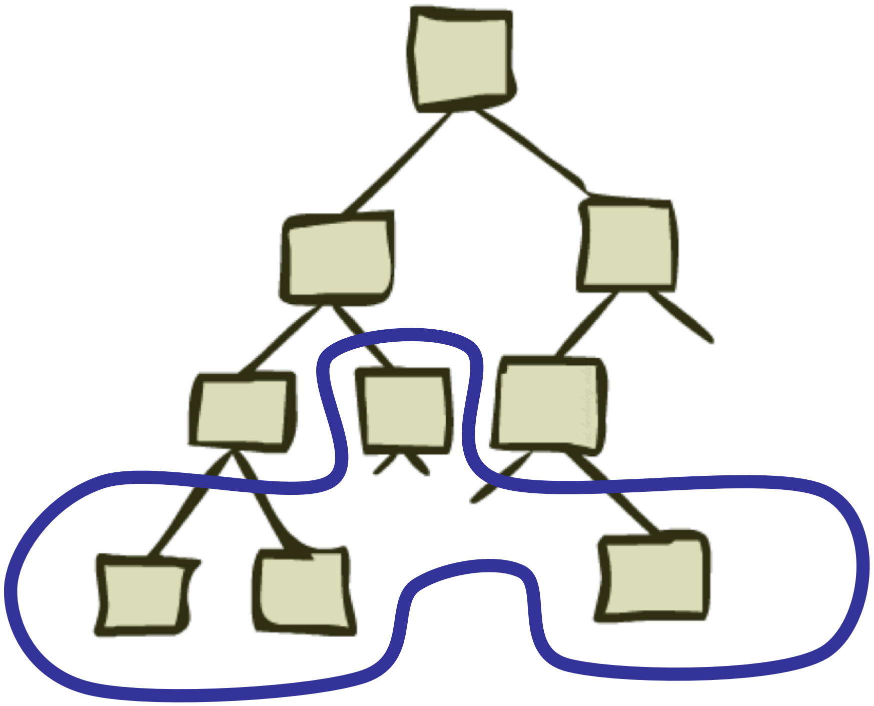
How big is its search tree (from S)?



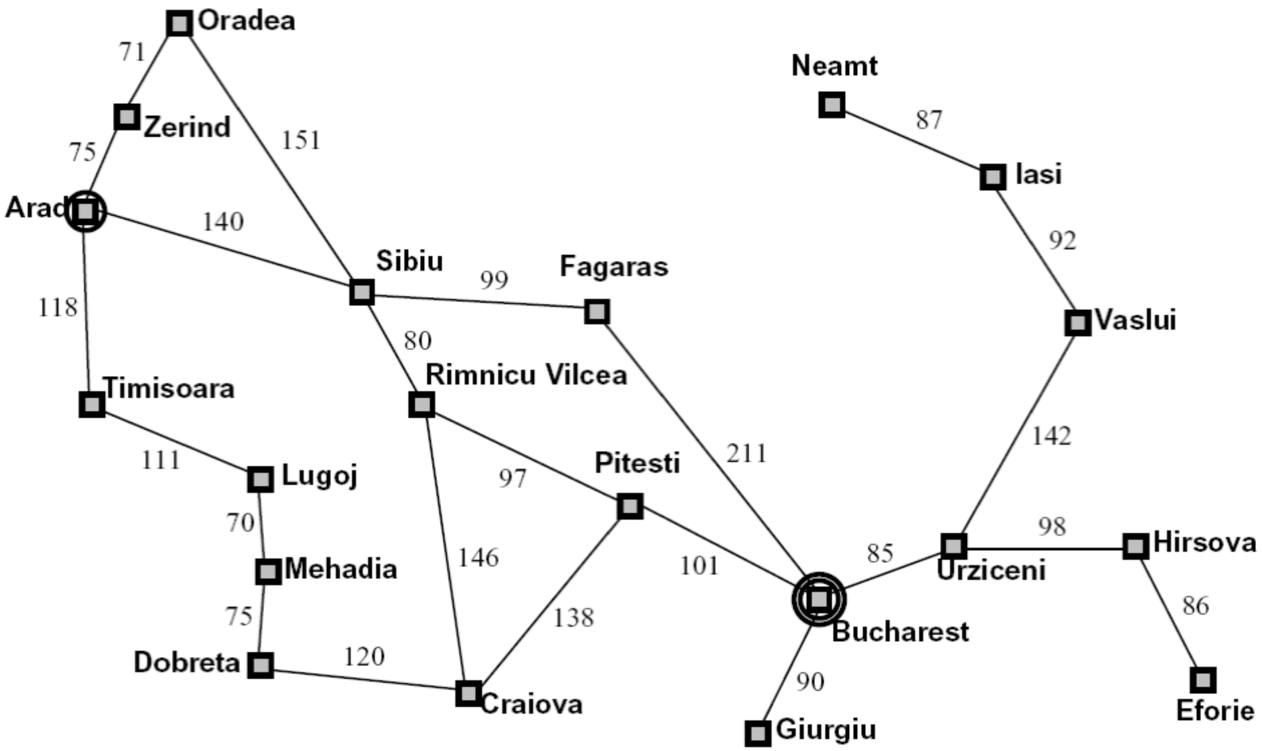
∞

Important: Lots of repeated structure in the search tree!

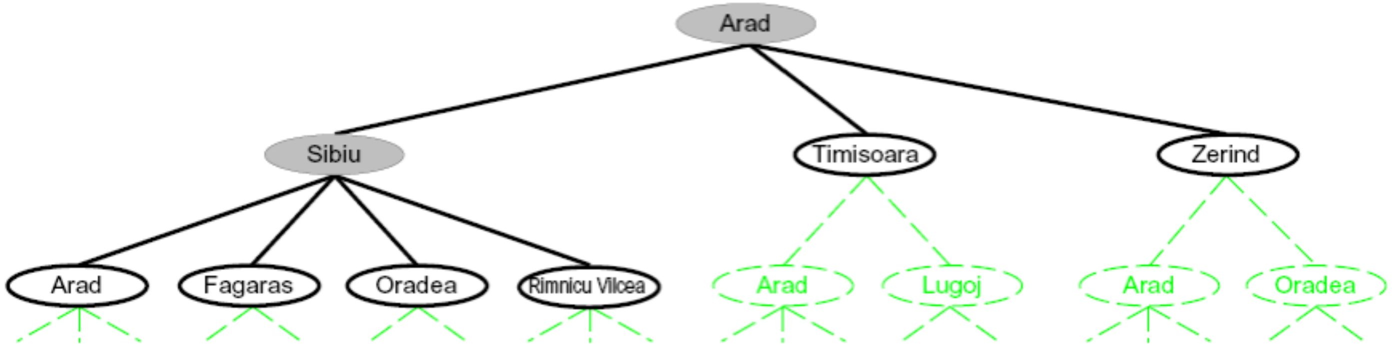
Tree Search



Search Example: Romania

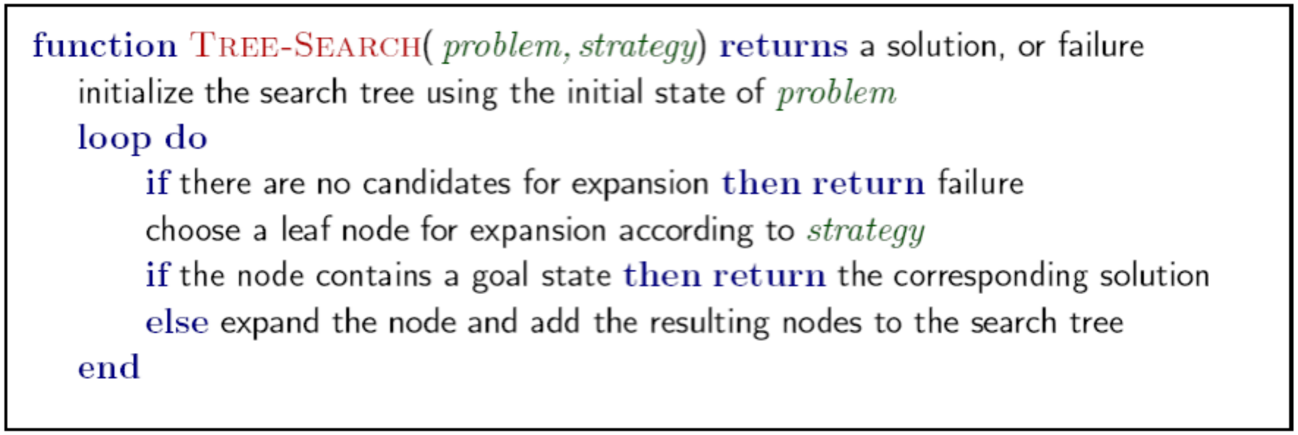


Searching with a Search Tree



- Search:
  - Expand out potential plans (tree nodes)
  - Maintain a fringe of partial plans under consideration
  - Try to expand as few tree nodes as possible

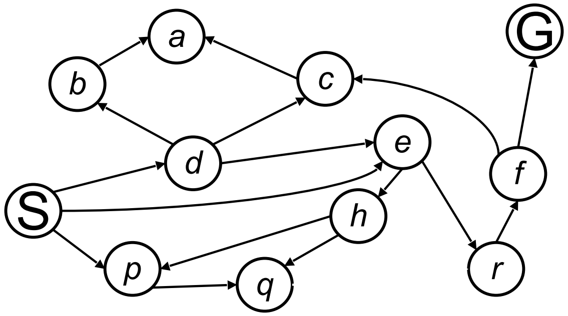
General Tree Search



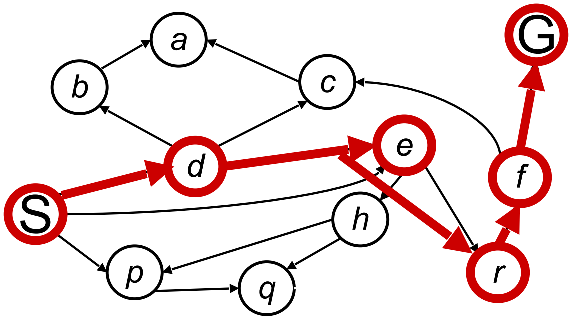
```
function TREE-SEARCH( problem, strategy) returns a solution, or failure
    initialize the search tree using the initial state of problem
    loop do
        if there are no candidates for expansion then return failure
        choose a leaf node for expansion according to strategy
        if the node contains a goal state then return the corresponding solution
        else expand the node and add the resulting nodes to the search tree
    end
```

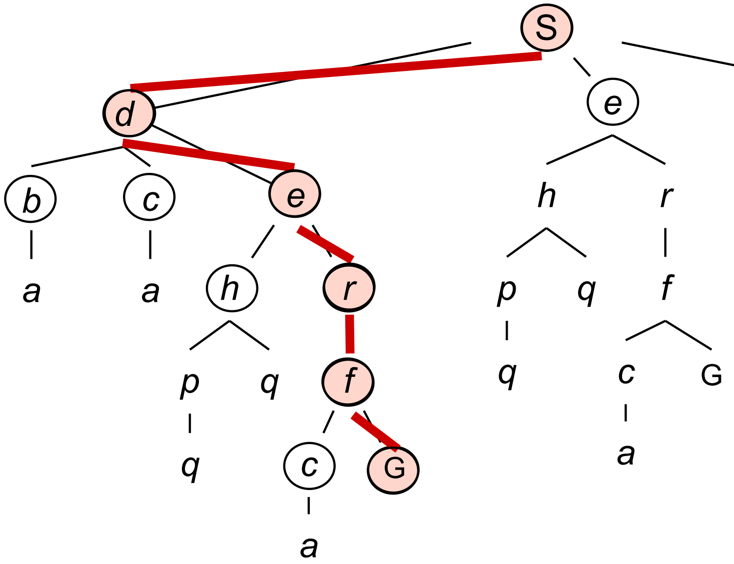
- Important ideas:
    - Fringe
    - Expansion
    - Exploration strategy
- Main question: which fringe nodes to explore?

Example: Tree Search



Example: Tree Search





~~s~~

~~s → d~~

s → e

s → p

s → d → b

~~s → d → c~~

s → d → e

~~s → d → e → h~~

~~s → d → e → r~~

~~s → d → e → r → f~~

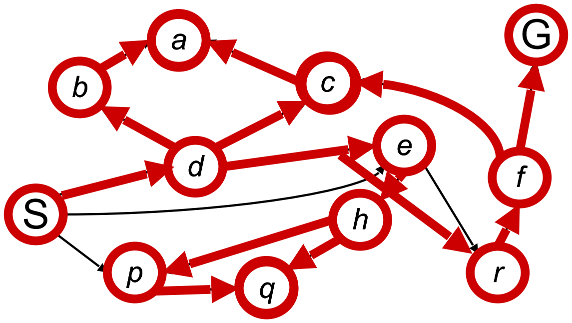
s → d → e → r → f → c

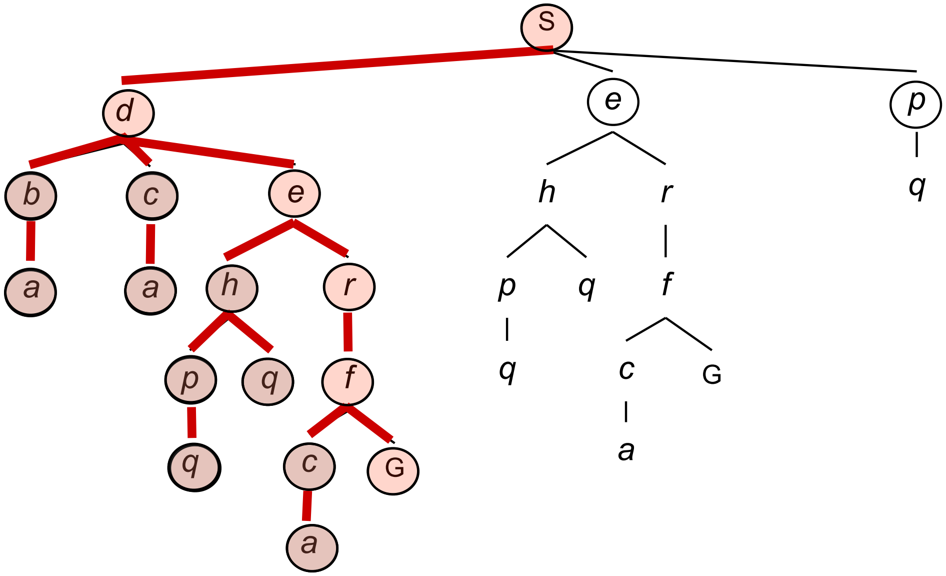
Depth-First Search

Depth-First Search

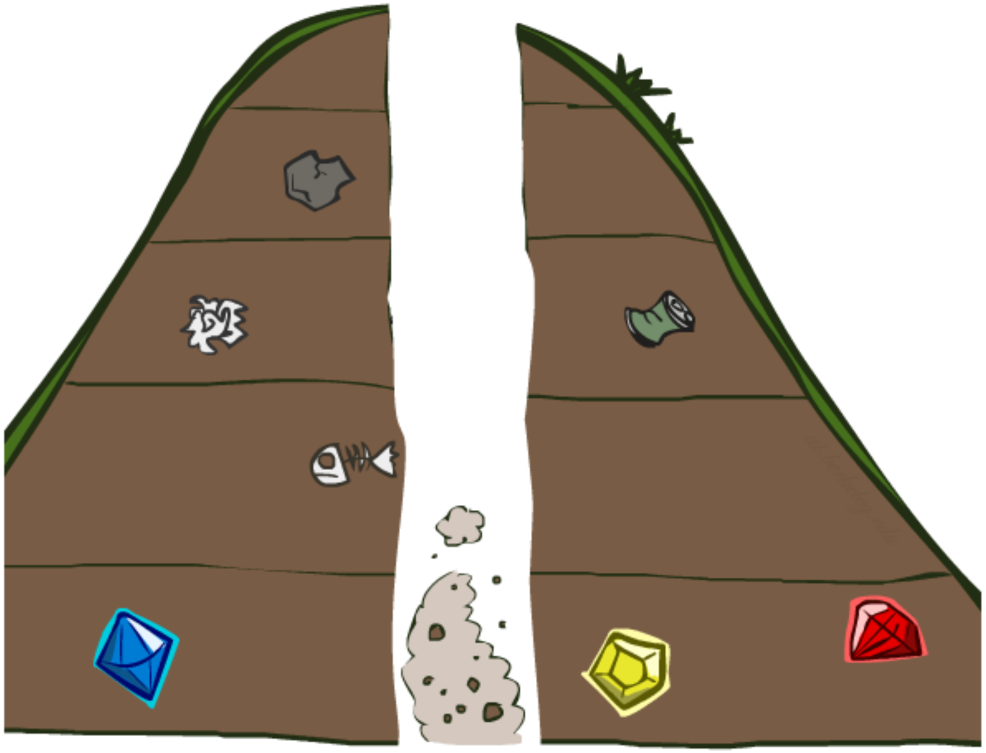
Strategy: expand a deepest node first

Implementation: Fringe is a LIFO stack





Search Algorithm Properties

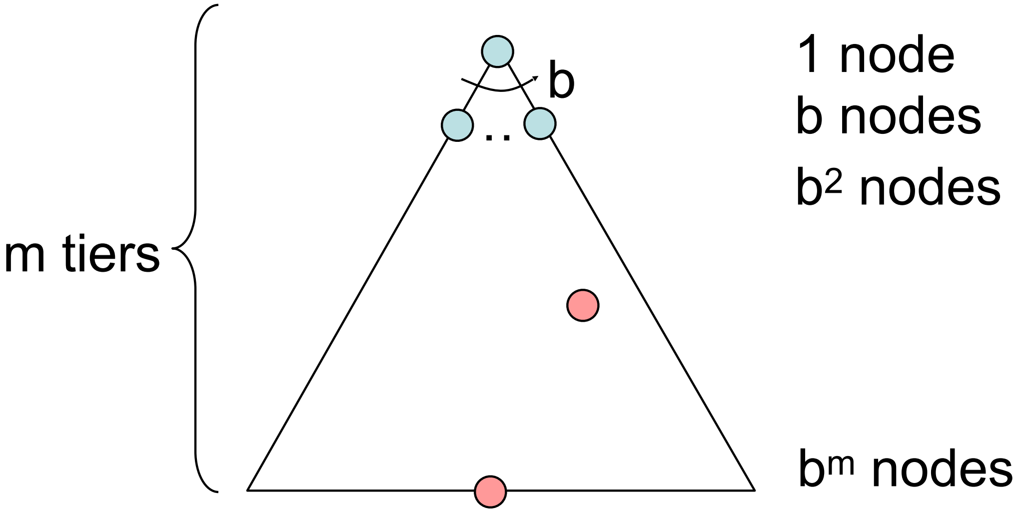


Search Algorithm Properties

- Complete: Guaranteed to find a solution if one exists?
- Optimal: Guaranteed to find the least cost path?
- Time complexity?
- Space complexity?

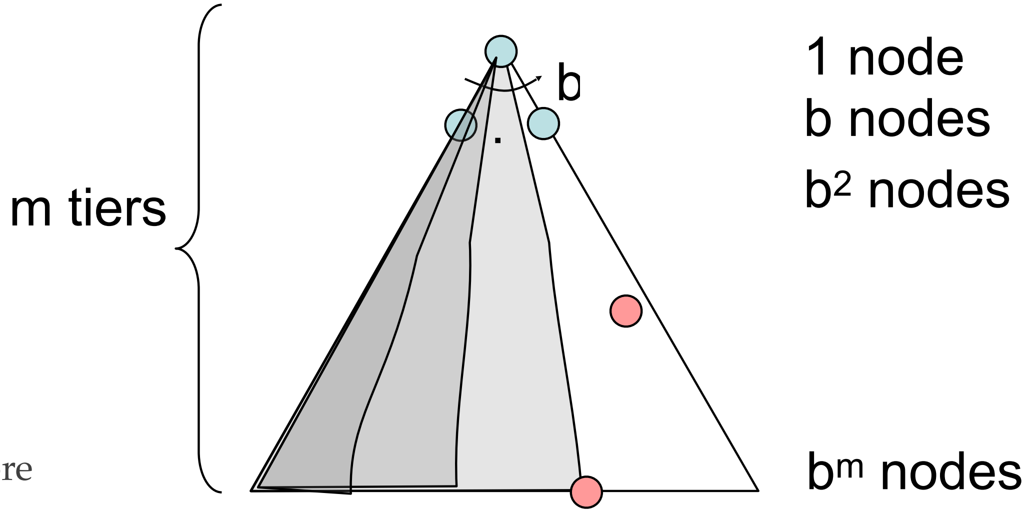
- Cartoon of search tree:
  - b is the branching factor
  - m is the maximum depth
  - solutions at various depths

- Number of nodes in entire tree?
  - $1 + b + b^2 + \ldots. b^m = O(b^m)$

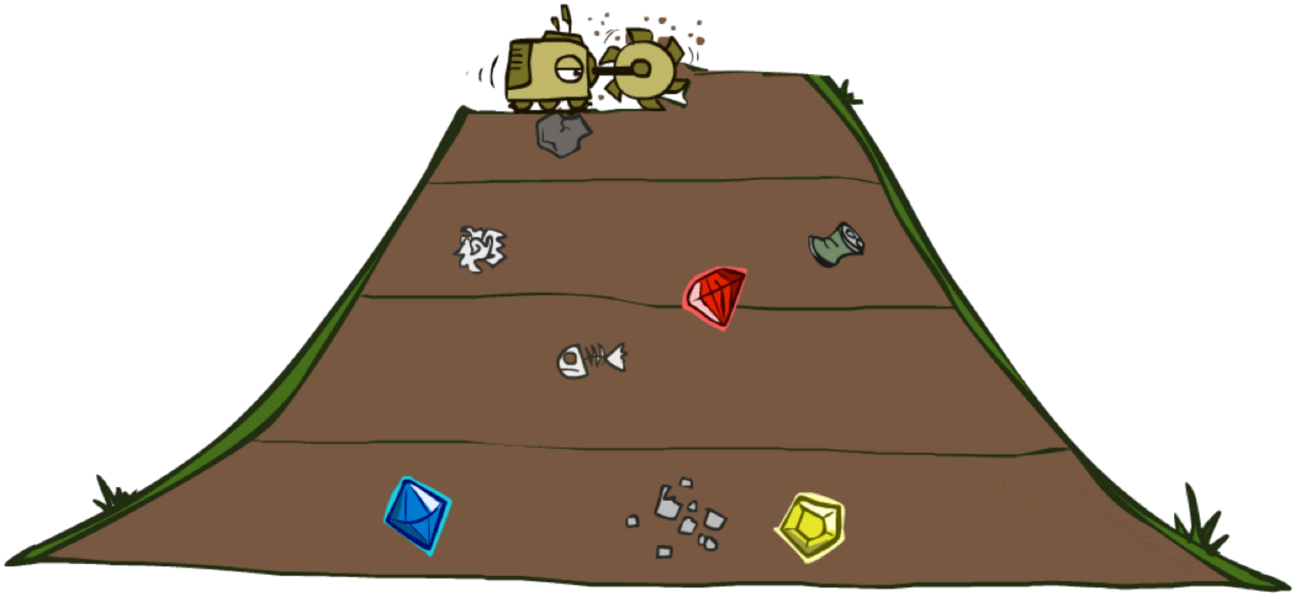


Depth-First Search (DFS) Properties

- What nodes DFS expand?
  - Some left prefix of the tree.
  - Could process the whole tree!
  - If m is finite, takes time $O(b^m)$
- How much space does the fringe take?
  - Only has siblings on path to root, so $O(bm)$
- Is it complete?
  - m could be infinite, so only if we prevent cycles (more later)
- Is it optimal?
  - No, it finds the “leftmost” solution, regardless of depth or cost



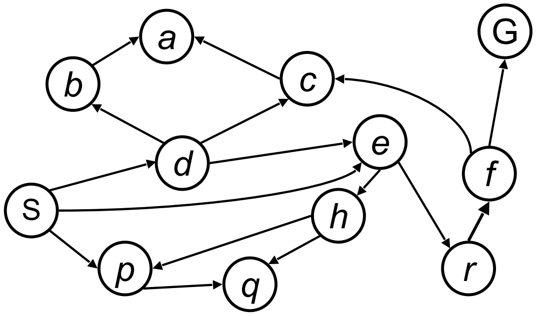
Breadth-First Search



Breadth-First Search

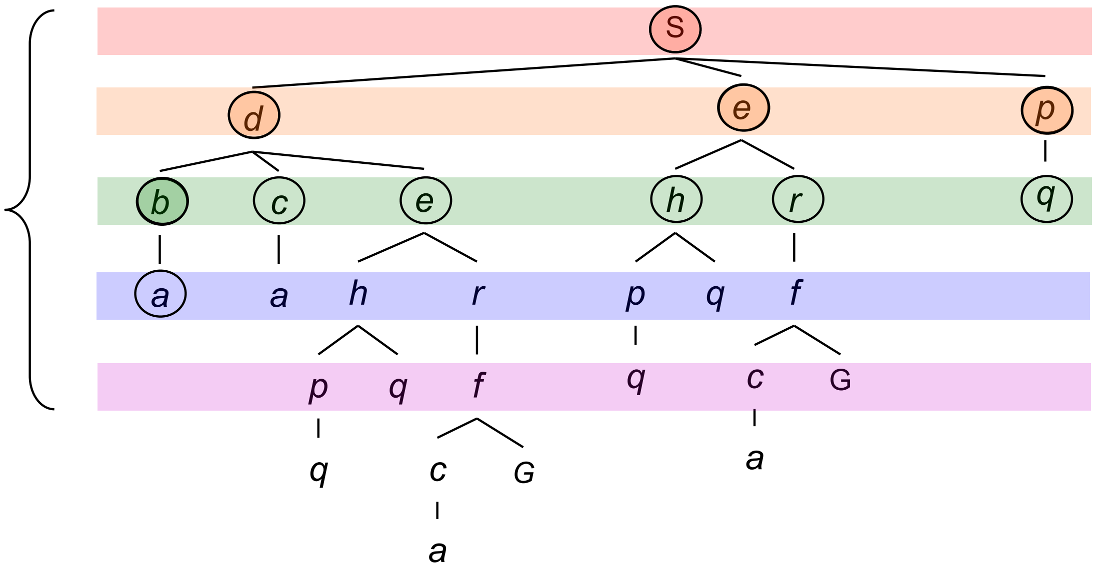
Strategy: expand a shallowest node first

Implementation: Fringe is a FIFO queue



Search

Tiers



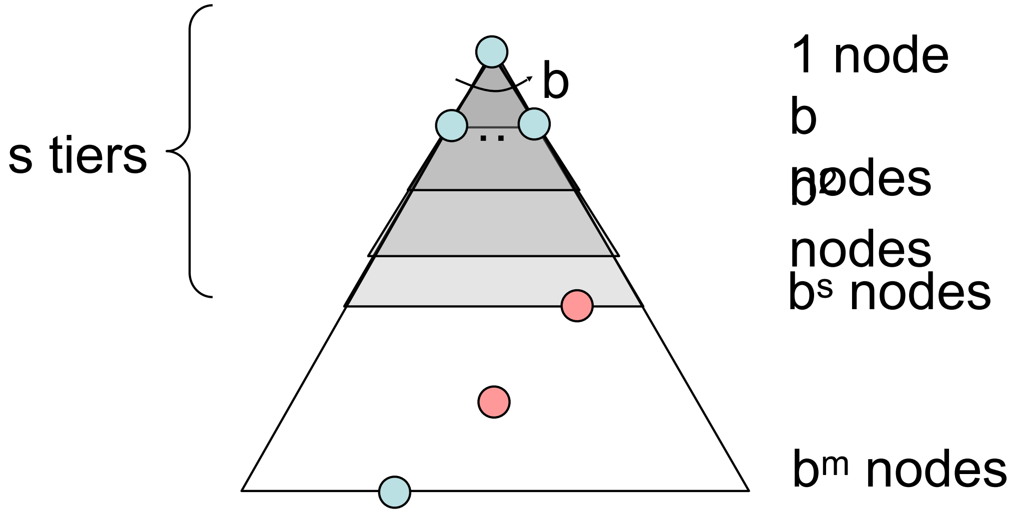
Breadth-First Search (BFS) Properties

- What nodes does BFS expand?
  - Processes all nodes above shallowest solution
  - Let depth of shallowest solution be s
  - Search takes time $O(b^s)$

- How much space does the fringe take?
  - Has roughly the last tier, so $O(b^s)$

- Is it complete?
  - s must be finite if a solution exists, so yes!

- Is it optimal?
  - Only if costs are all 1 (more on costs later)

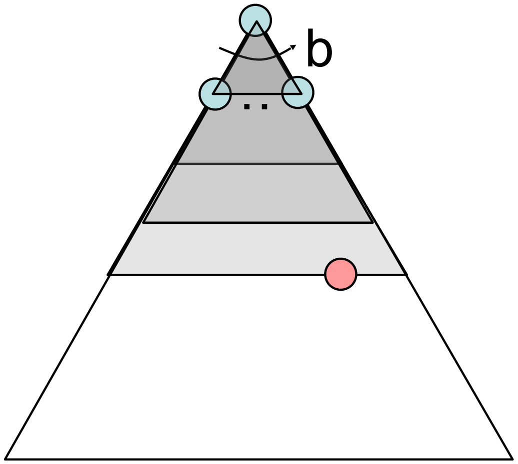


Video of Demo Maze Water DFS/BFS (part 1)

Video of Demo Maze Water DFS/BFS (part 2)

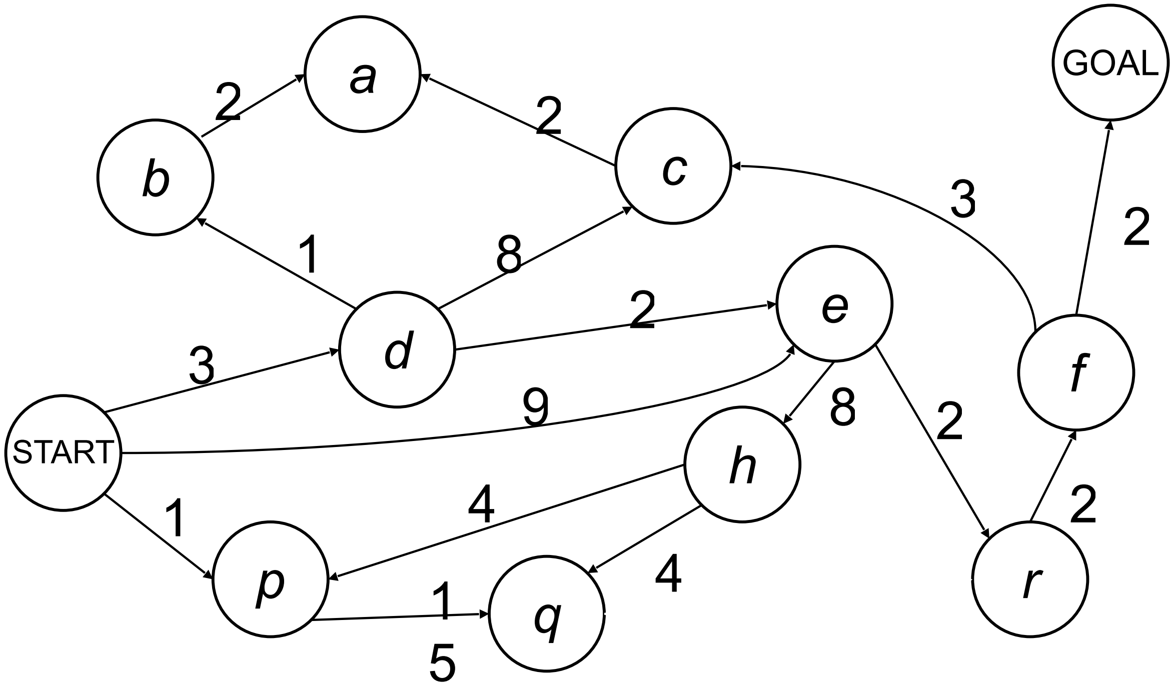
Iterative Deepening

- Idea: get DFS's space advantage with BFS's time / shallow-solution advantages
  - Run a DFS with depth limit 1. If no solution…
  - Run a DFS with depth limit 2. If no solution…
  - Run a DFS with depth limit 3. …..



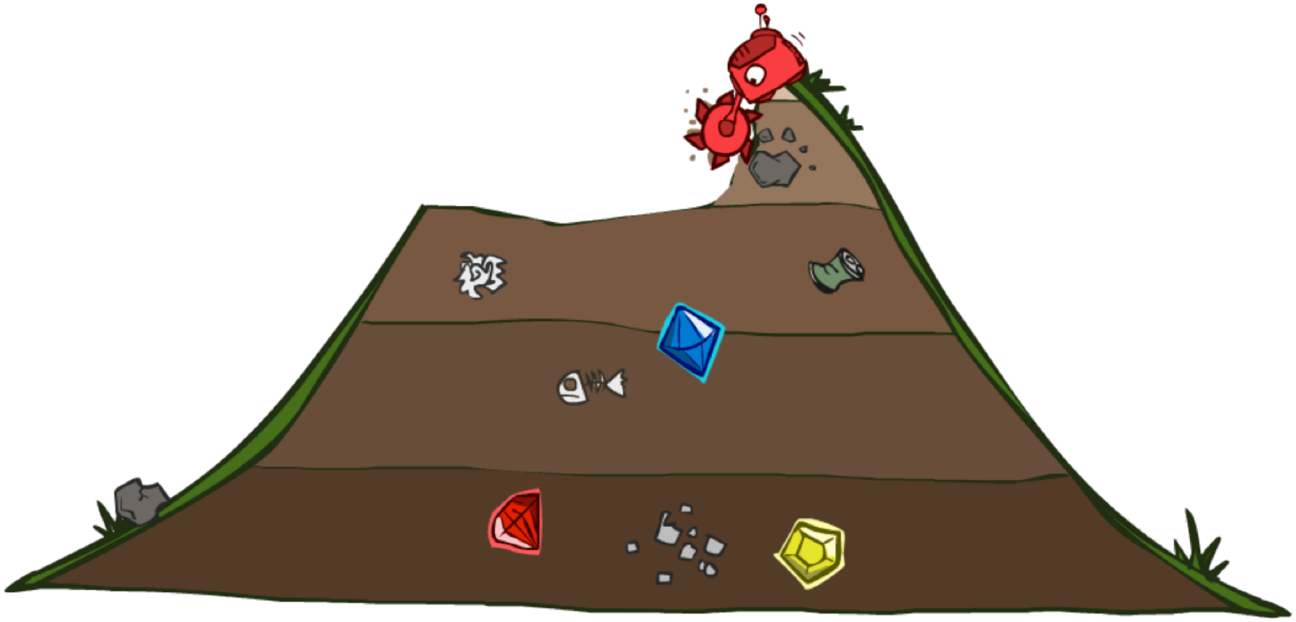
- Isn't that wastefully redundant?
  - Generally most work happens in the lowest level searched, so not so bad!

Cost-Sensitive Search



BFS finds the shortest path in terms of number of actions.
It does not find the least-cost path. We will now cover
a similar algorithm which does find the least-cost path.

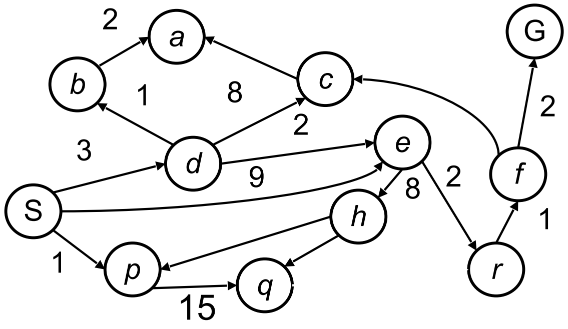
Uniform Cost Search



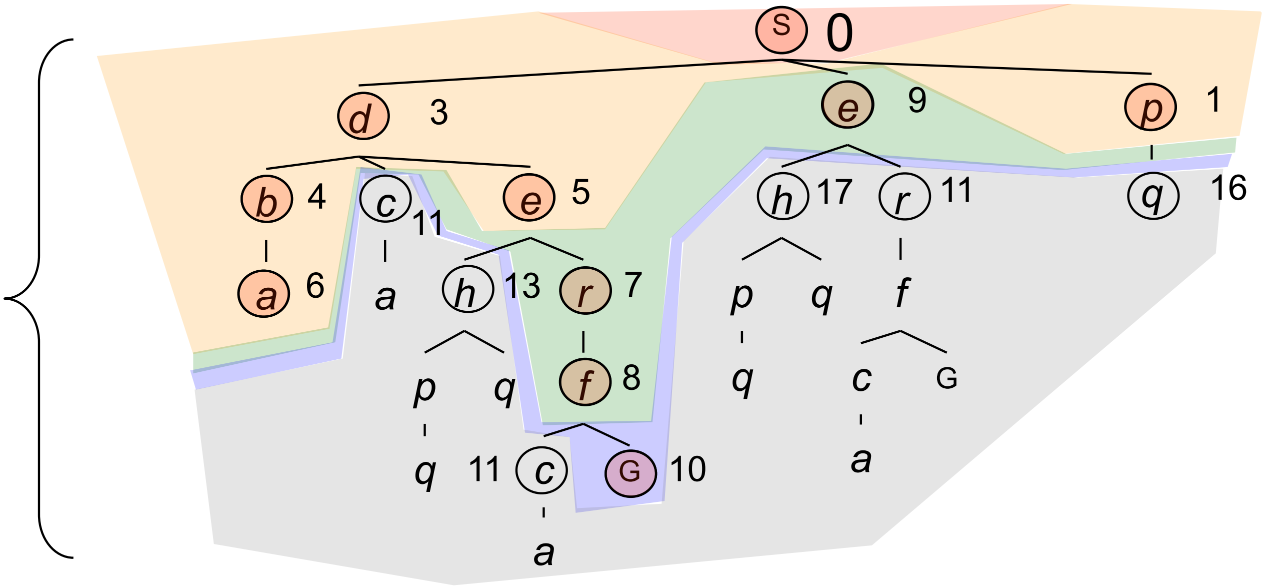
Uniform Cost Search

Strategy: expand a cheapest node first:

Fringe is a priority queue (priority: cumulative cost)

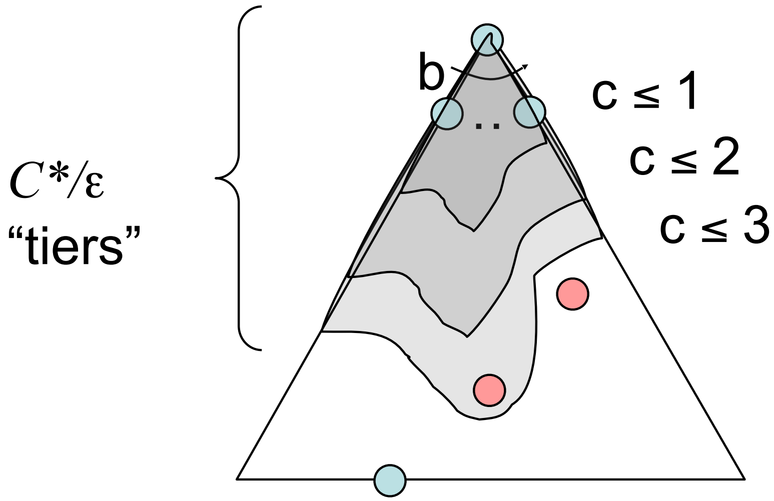


Cost contours



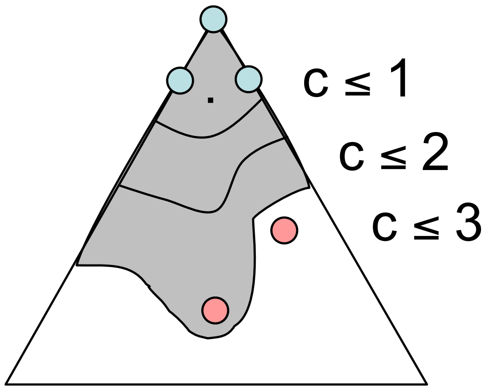
Uniform Cost Search (UCS) Properties

- What nodes does UCS expand?
  - Processes all nodes with cost less than cheapest solution!
  - If that solution costs C^* and arcs cost at least ε , then the "effective depth" is roughly C^*/ε
  - Takes time O($b^{C^*/\varepsilon}$) (exponential in effective depth)
- How much space does the fringe take?
  - Has roughly the last tier, so O($b^{C^*/\varepsilon}$)
- Is it complete?
  - Assuming best solution has a finite cost and minimum arc cost is positive, yes!
- Is it optimal?
  - Yes! (Proof via A*)

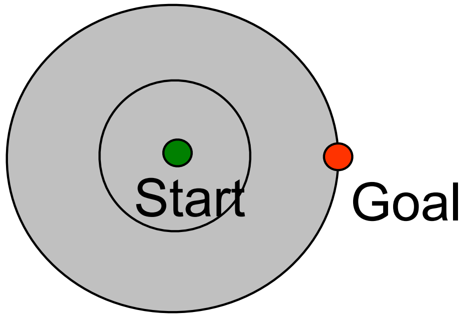


Uniform Cost Issues

- Remember: UCS explores increasing cost contours



- The good: UCS is complete and optimal!
- The bad:
  - Explores options in every "direction"
  - No information about goal location



- We'll fix that soon!

Video of Demo Empty UCS

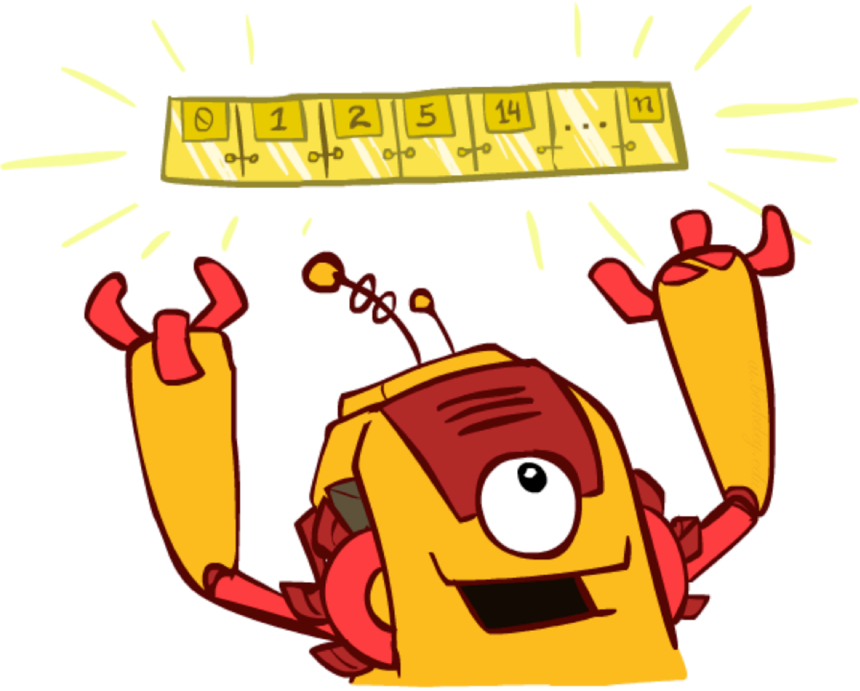
Demo Maze with Deep/Shallow Water --- DFS, BFS, or UCS? (part 1)

Demo Maze with Deep/Shallow Water --- DFS, BFS, or UCS? (part 2)

Demo Maze with Deep/Shallow Water --- DFS, BFS, or UCS? (part 3)

The One Queue

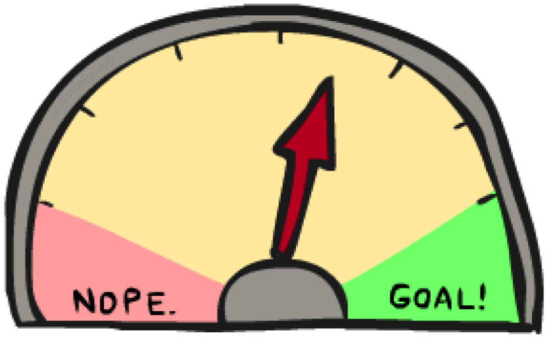
- All these search algorithms are the same except for fringe strategies
    - DFS: Fringe is a Stack
    - BFS: Fringe is a Queue
    - UCS: Fringe is a PriorityQueue
    - Can even code one implementation th takes a variable queuing object



Up next: Informed Search

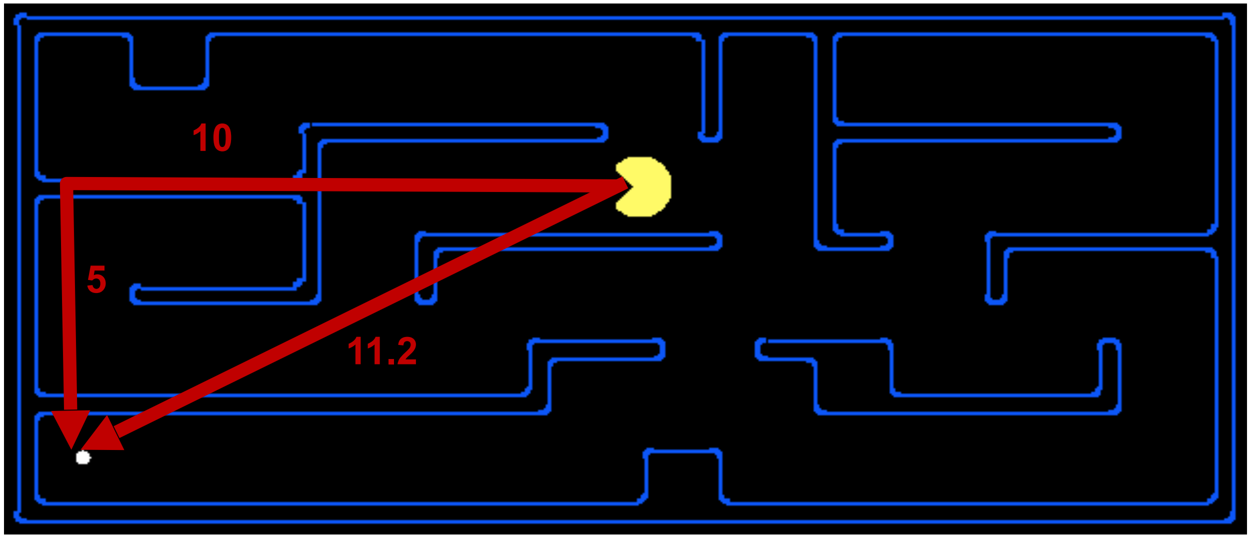
- Uninformed Search
  - DFS
  - BFS
  - UCS

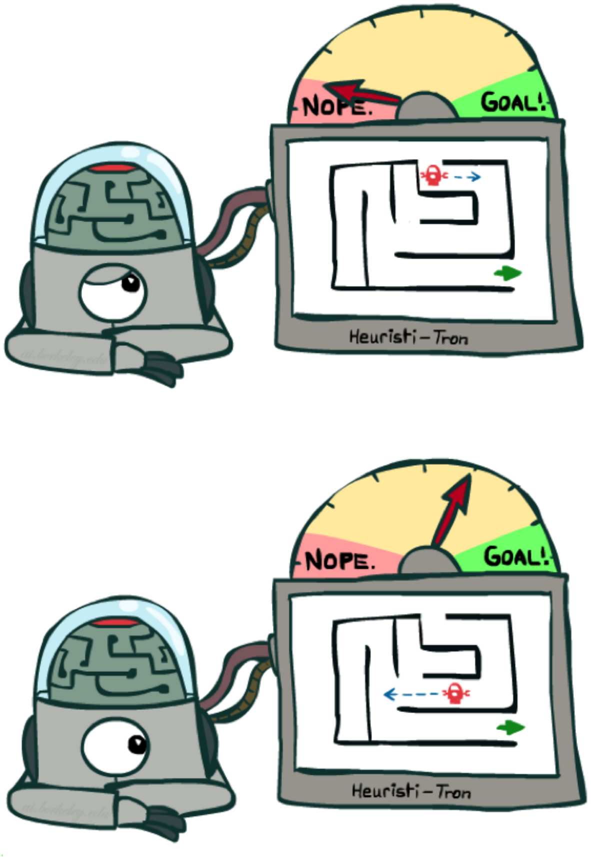
- Informed Search (Heuristics)
  - Greedy Search
  - A* Search



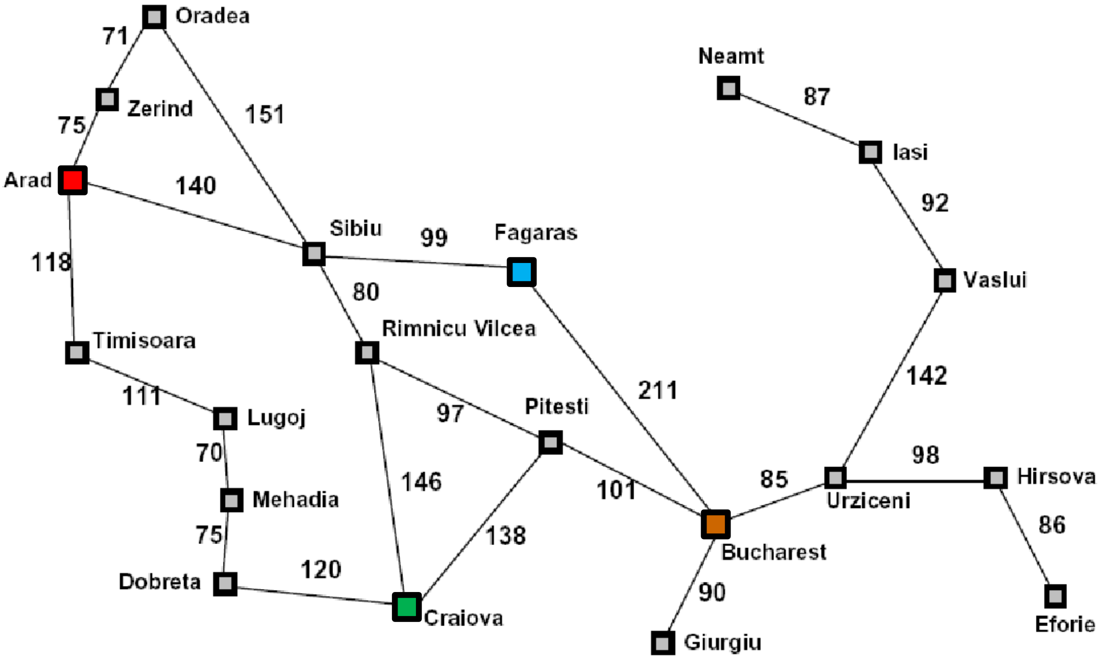
Search Heuristics

- A heuristic is:
  - A function that *estimates* how close a state is to a goal
  - Designed for a particular search problem
  - Pathing?
  - Examples: Manhattan distance, Euclidean distance





Example: Heuristic Function



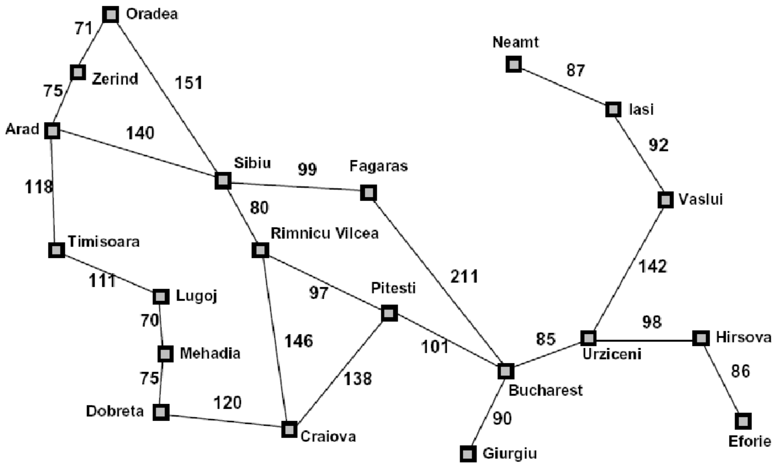
Straight−line distance
to Bucharest

Arad	366
Bucharest	0
Craiova	160
Dobreta	242
Eforie	161
Fagaras	178
Giurgiu	77
Hirsova	151
Iasi	226
Lugoj	244
Mehadia	241
Neamt	234
Oradea	380
Pitesti	98
Rimnicu Vilcea	193
Sibiu	253
Timisoara	329
Urziceni	80
Vaslui	199
Zerind	374

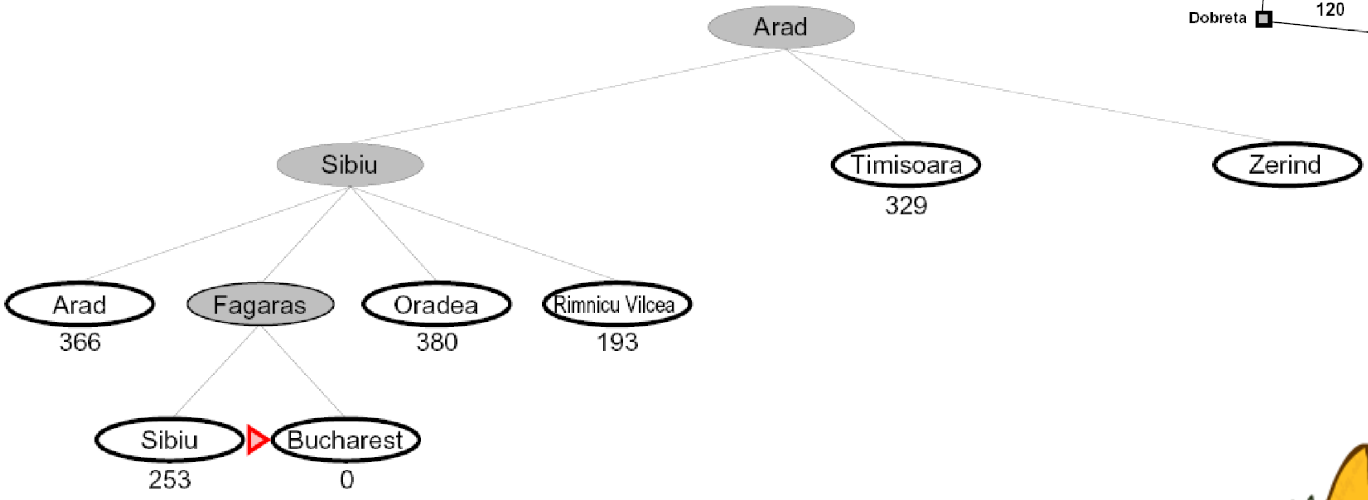
h(x)

Greedy Search

Greedy Search



- Expand the node that seems closest…
  - Move to smallest heuristic value

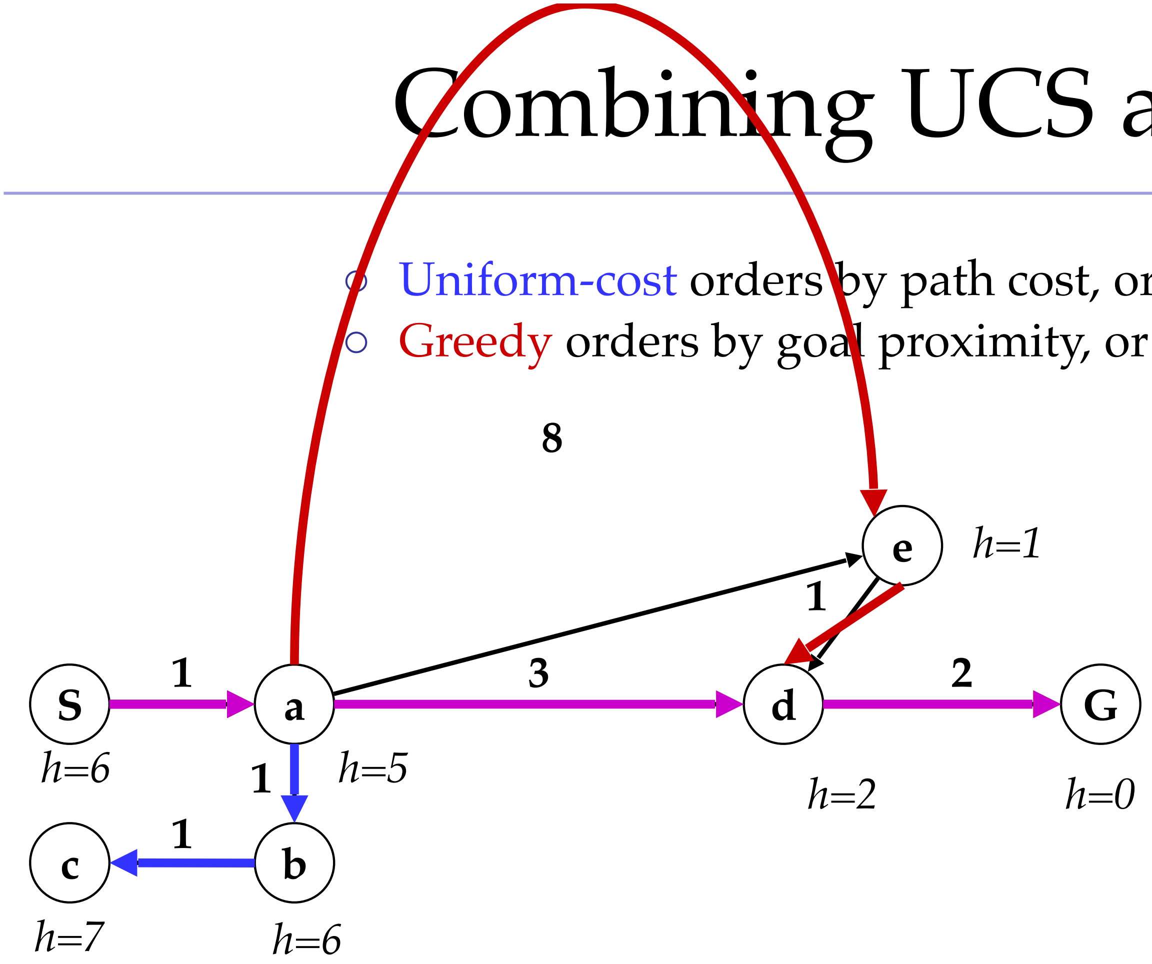


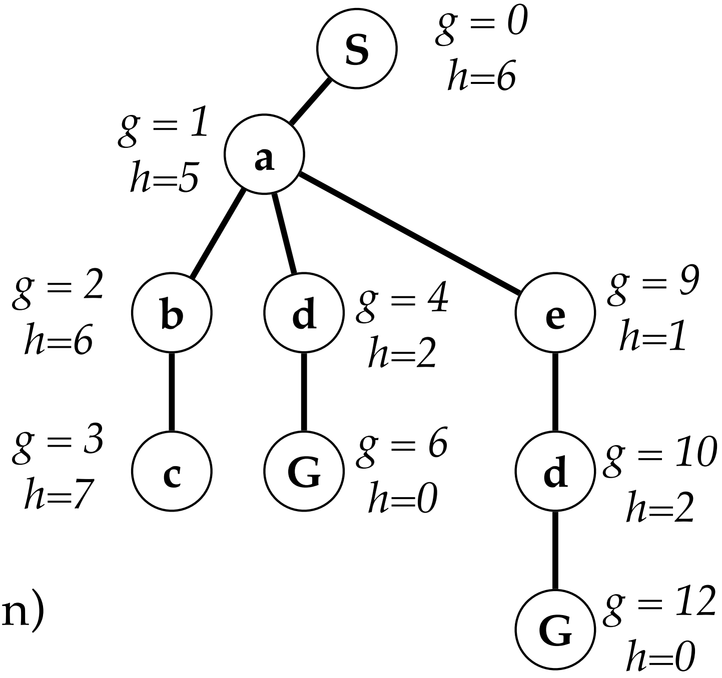
- Is it optimal?
  - No. Resulting path to Bucharest is not the shortest!

A* Search

Combining UCS and Greedy

- Uniform-cost orders by path cost, or *backward cost* g(n)
- Greedy orders by goal proximity, or *forward cost* h(n)



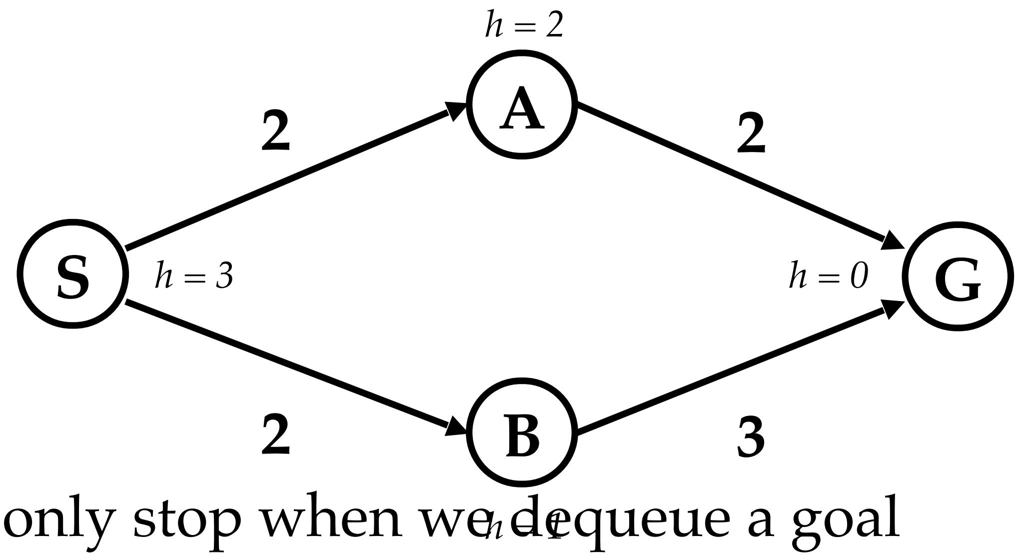


- A* Search orders by the sum: f(n) = g(n) + h(n)

Example: Teg Grenager

When should A* terminate?

- Should we stop when we enqueue a goal?



- No: only stop when we dequeue a goal

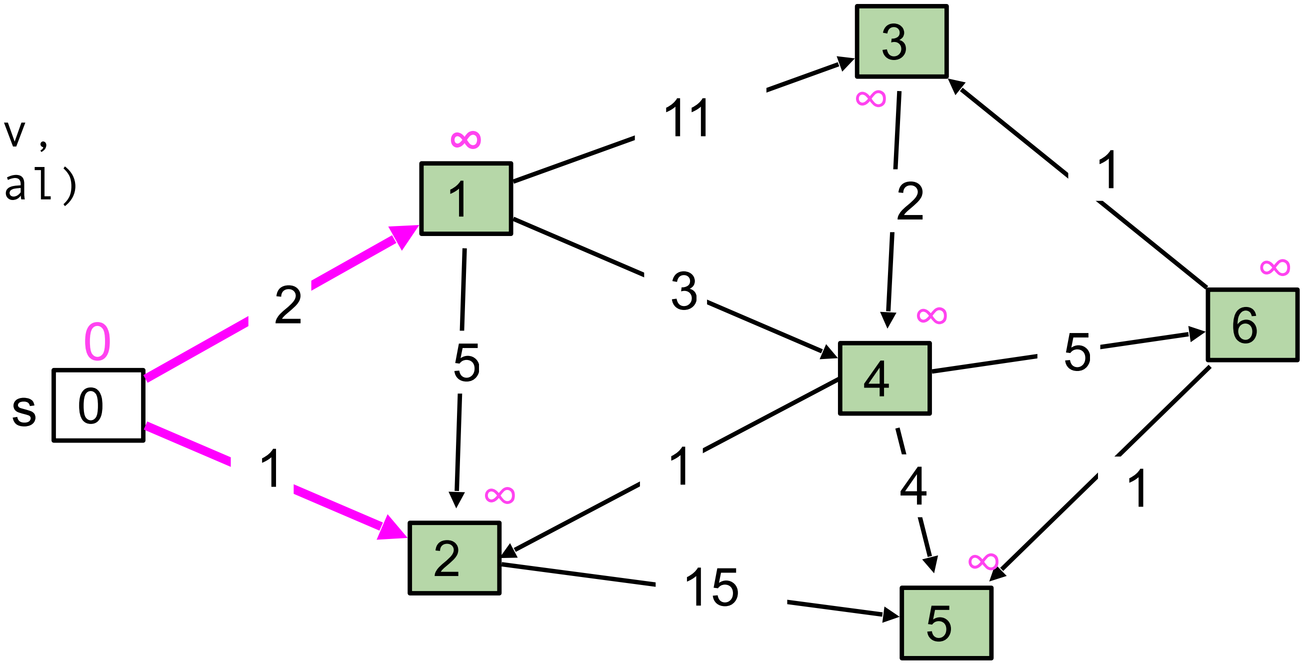
	g	h	+
~~S~~	~~0~~	~~3~~	~~3~~
~~S->A~~	~~2~~	~~2~~	~~4~~
~~S->B~~	~~2~~	~~1~~	~~3~~
S->B->G	5	0	5
S->A->G	4	0	4

A* Demo, with s = 0, goal = 6. (Credit: Josh Hug)

Insert all vertices into fringe PQ, storing vertices in order of d(source, v) + h(v, goal).

Repeat: Remove best vertex v from PQ, and relax all edges pointing from v.

#	distTo	edgeTo	h(v, goal)
0	0	-	1
1	∞	-	3
2	∞	-	15
3	∞	-	1
4	∞	-	5
5	∞	-	∞
6	∞	-	0



h(v, goal) is arbitrary. In this example, it's the min weight edge out of each vertex.

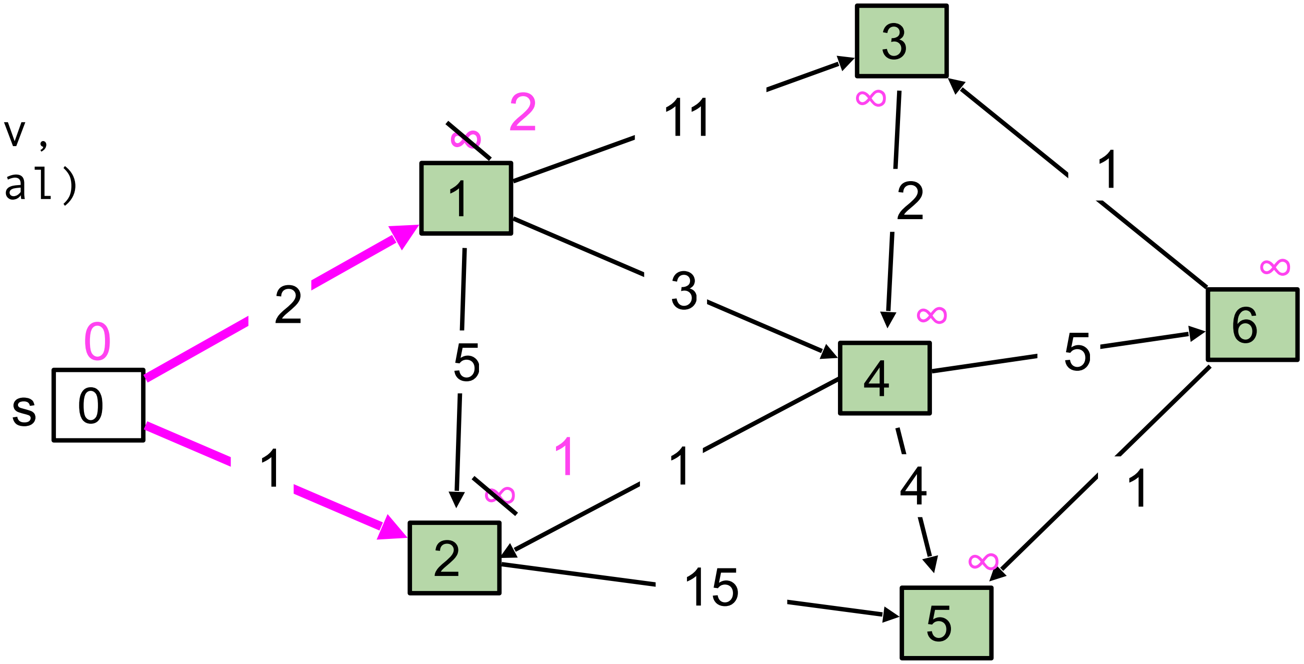
`Fringe: [(1: ∞), (2: ∞), (3: ∞), (4: ∞), (5: ∞), (6: ∞)]`

A* Demo, with s = 0, goal = 6.

Insert all vertices into fringe PQ, storing vertices in order of d(source, v) + h(v, goal).

Repeat: Remove best vertex v from PQ, and relax all edges pointing from v.

#	distTo	edgeTo	h(v, goal)
0	0	-	1
1	2	0	3
2	1	0	15
3	∞	-	1
4	∞	-	5
5	∞	-	∞
6	∞	-	0



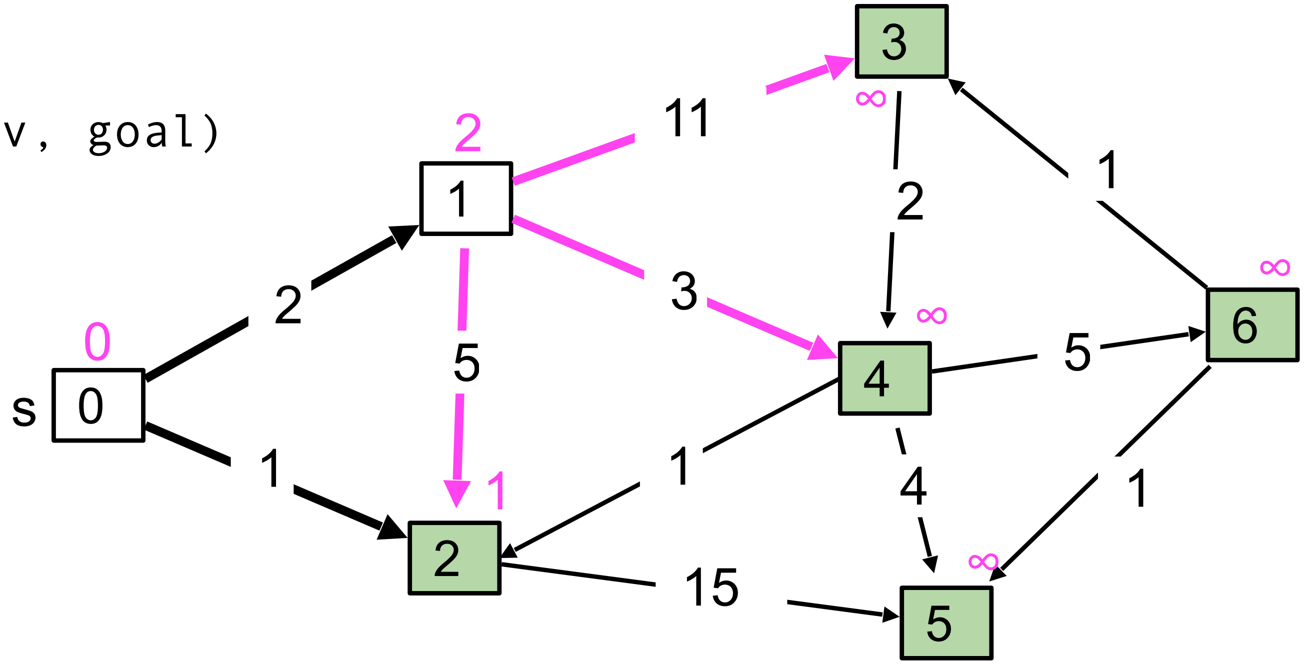
Fringe: [(1: 5), (2: 16), (3: ∞), (4: ∞), (5: ∞), (6: ∞)]

A* Demo, with s = 0, goal = 6.

Insert all vertices into fringe PQ, storing vertices in order of d(source, v) + h(v, goal).

Repeat: Remove best vertex v from PQ, and relax all edges pointing from v.

#	distTo	edgeTo	h(v, goal)
0	0	-	1
1	2	0	3
2	1	0	15
3	∞	-	2
4	∞	-	1
5	∞	-	∞
6	∞	-	0



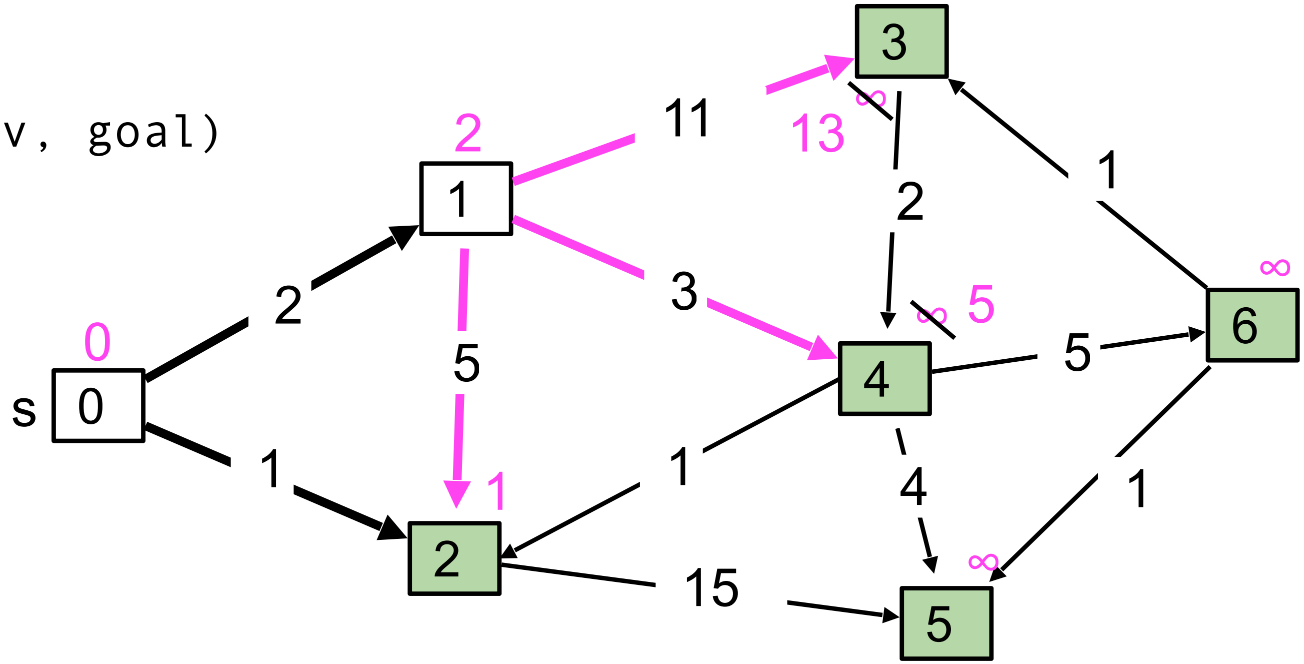
Fringe: [(2: 16), (3: ∞), (4: ∞), (5: ∞), (6: ∞)]

A* Demo, with s = 0, goal = 6.

Insert all vertices into fringe PQ, storing vertices in order of d(source, v) + h(v, goal).

Repeat: Remove best vertex v from PQ, and relax all edges pointing from v.

#	distTo	edgeTo	h(v, goal)
0	0	-	1
1	2	0	3
2	1	0	15
3	13	1	2
4	5	1	1
5	∞	-	∞
6	∞	-	0



```
Fringe: [(4: 6), (3: 15), (2: 16), (5: ∞), (6: ∞)]
```

Which vertex is removed

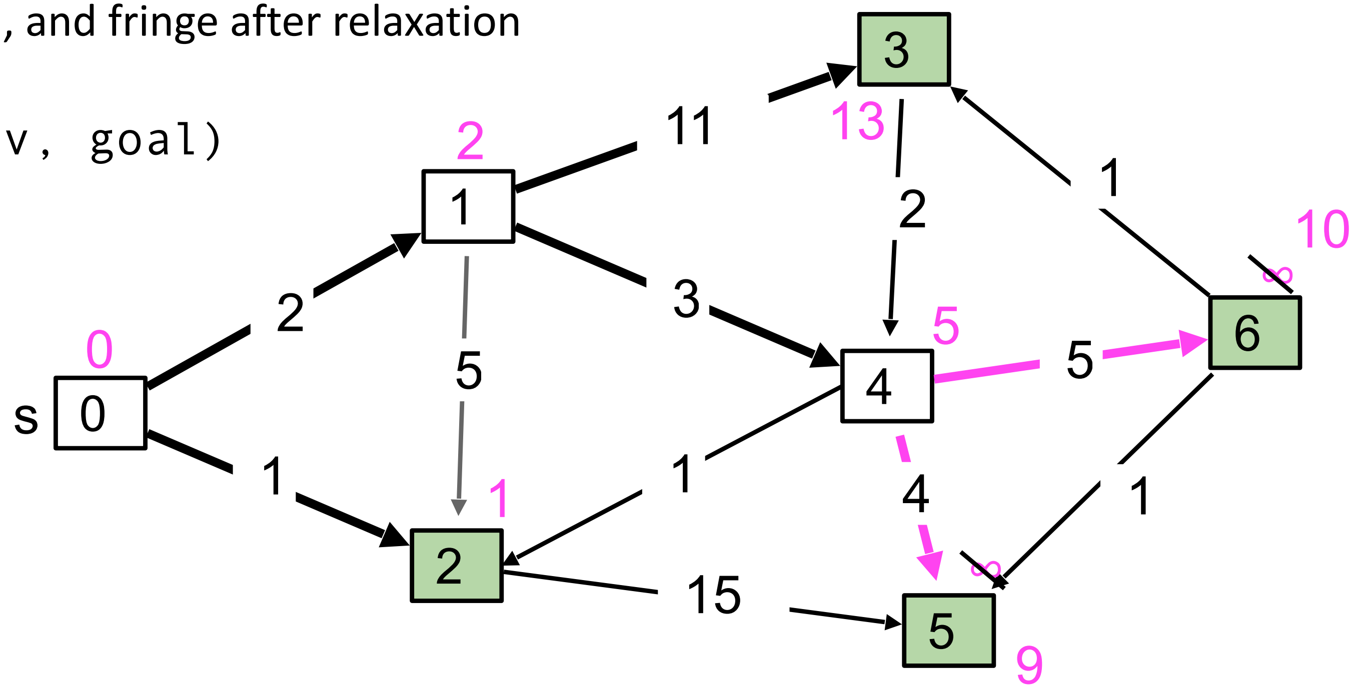
A* Demo, with s = 0, goal = 6.

Insert all vertices into fringe PQ, storing vertices in order of d(source, v) + h(v, goal).

Repeat: Remove best vertex v from PQ, and relax all edges pointing from v.

- Give distTo, edgeTo, h(v, goal), and fringe after relaxation

#	distTo	edgeTo	h(v, goal)
0	0	-	1
1	2	0	3
2	1	0	15
3	13	1	2
4	5	1	1
5	9	4	∞
6	10	4	0



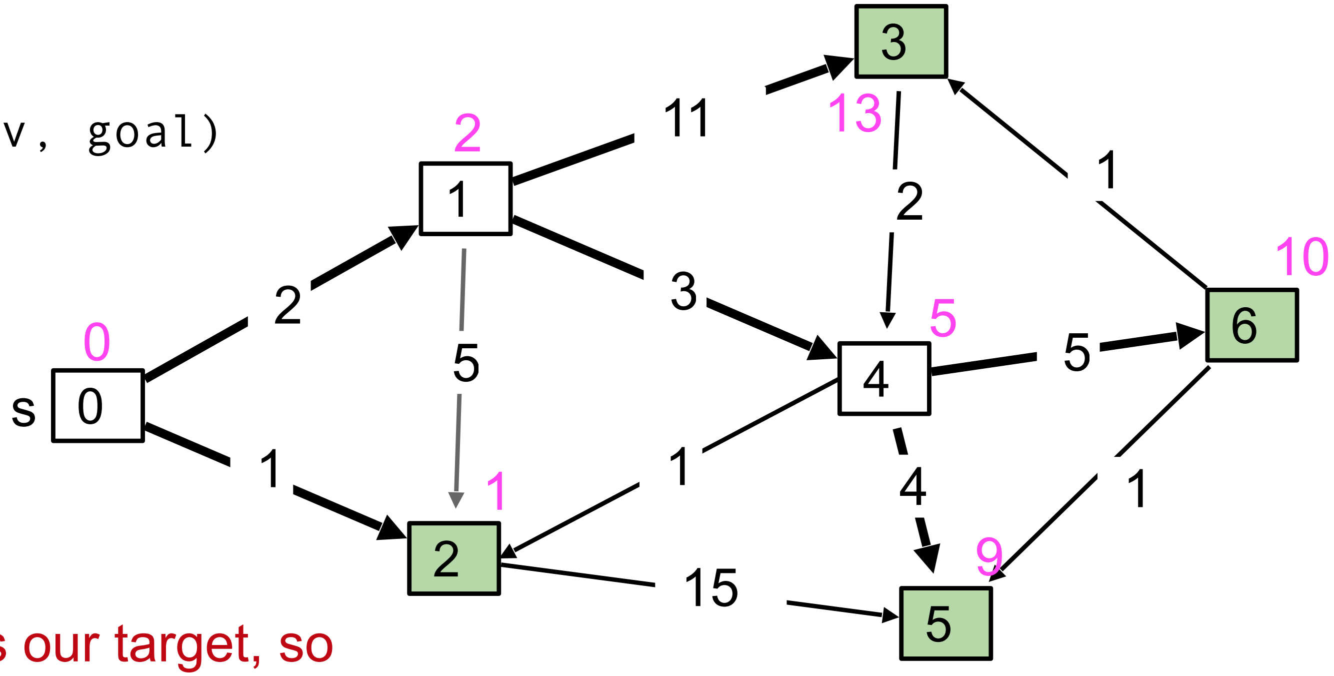
`Fringe: [(6: 10), (3: 15), (2: 16), (5: ∞)]`

A* Demo, with s = 0, goal = 6.

Insert all vertices into fringe PQ, storing vertices in order of d(source, v) + h(v, goal).

Repeat: Remove best vertex v from PQ, and relax all edges pointing from v.

#	distTo	edgeTo	h(v, goal)
0	0	-	1
1	2	0	3
2	1	0	15
3	13	1	2
4	5	1	1
5	9	4	∞
6	10	4	0



Next vertex to be dequeued is our target, so we’re done!

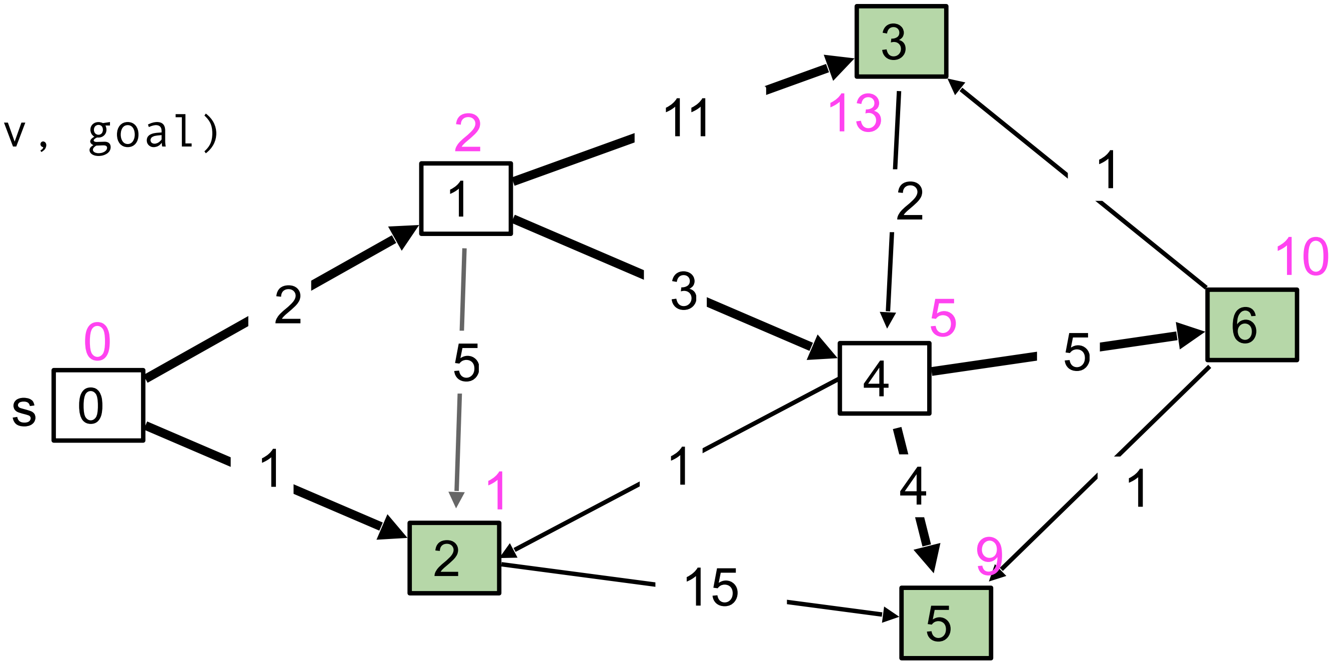
`Fringe: [(6: 10), (3: 15), (2: 16), (5: ∞)]`

A* Demo, with s = 0, goal = 6.

Insert all vertices into fringe PQ, storing vertices in order of d(source, v) + h(v, goal).

Repeat: Remove best vertex v from PQ, and relax all edges pointing from v.

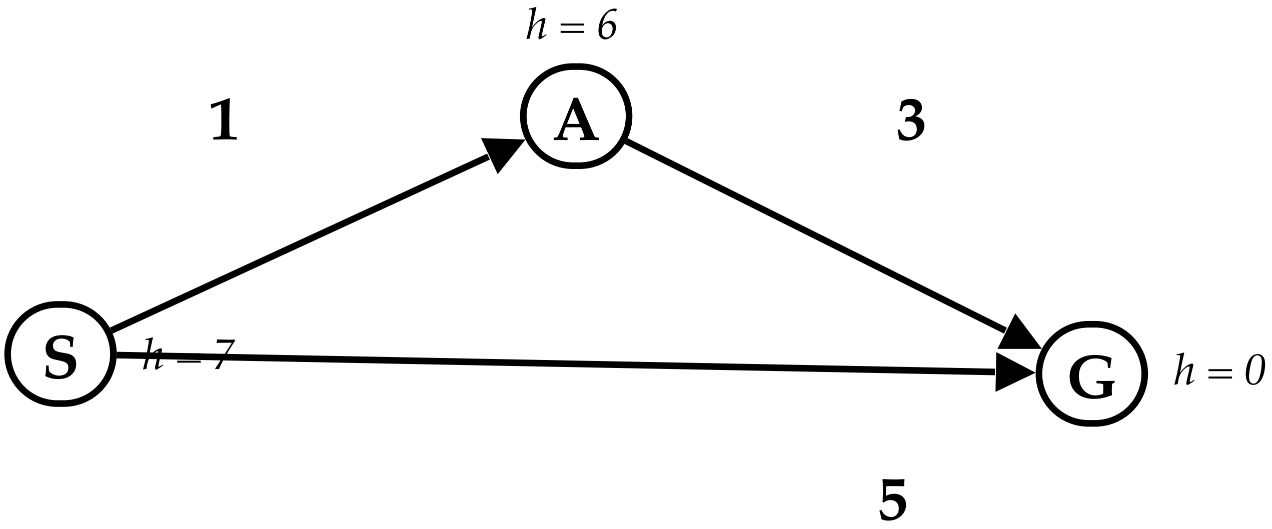
#	distTo	edgeTo	h(v, goal)
0	0	-	1
1	2	0	3
2	1	0	15
3	13	1	2
4	5	1	1
5	9	4	∞
6	10	4	0



Observations:

- Not every vertex got visited.
- Result is not a shortest paths tree for vertex zero (path to 3 is suboptimal!), but that’s OK because we only care about path to 6.

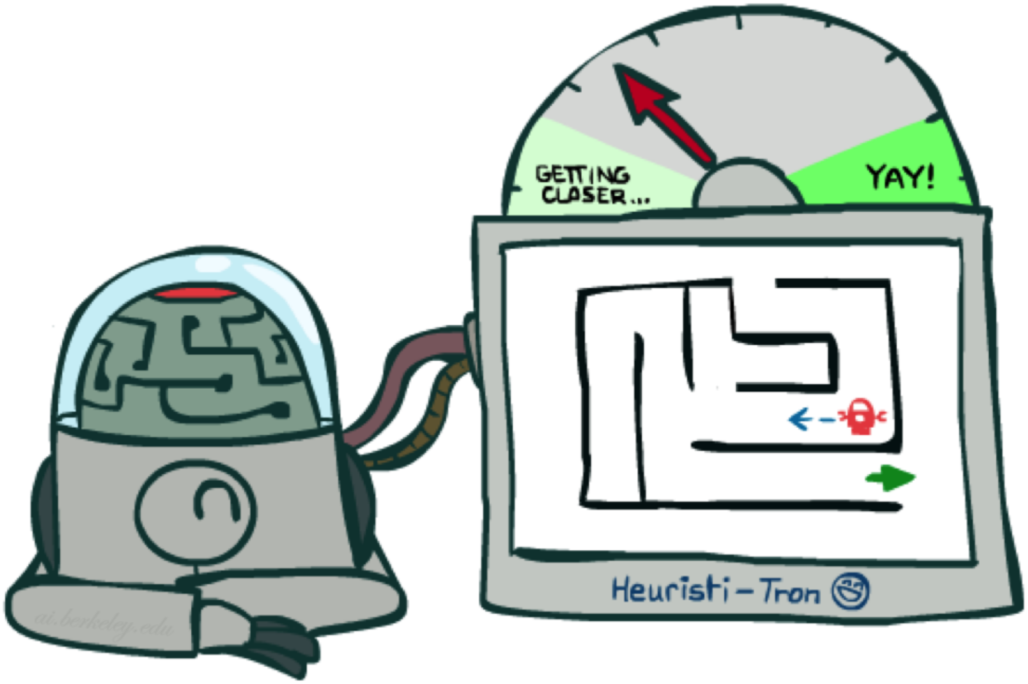
Is A* Optimal?



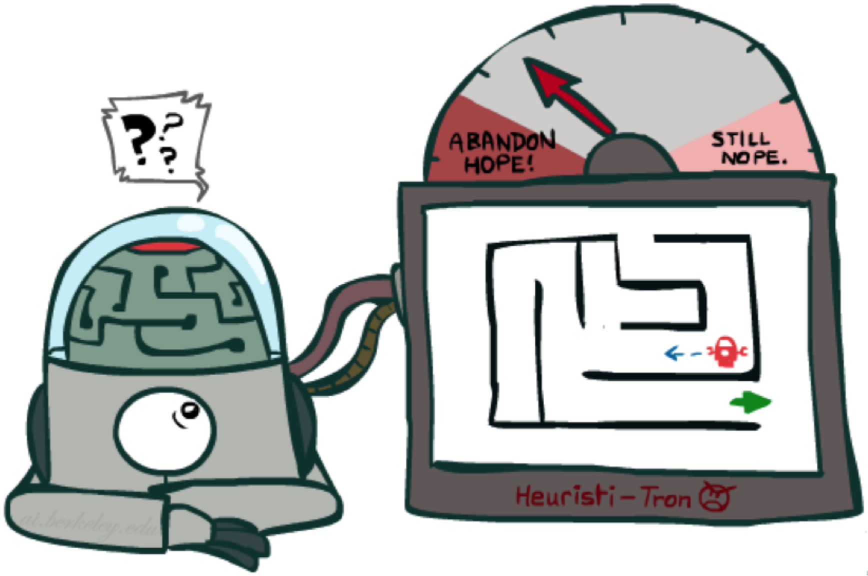
	g	h	+
~~S~~	~~0~~	~~7~~	~~7~~
S->A	1	6	7
S->G	5	0	5

- What went wrong?
- Actual bad goal cost < estimated good goal cost
- We need estimates to be less than actual costs!

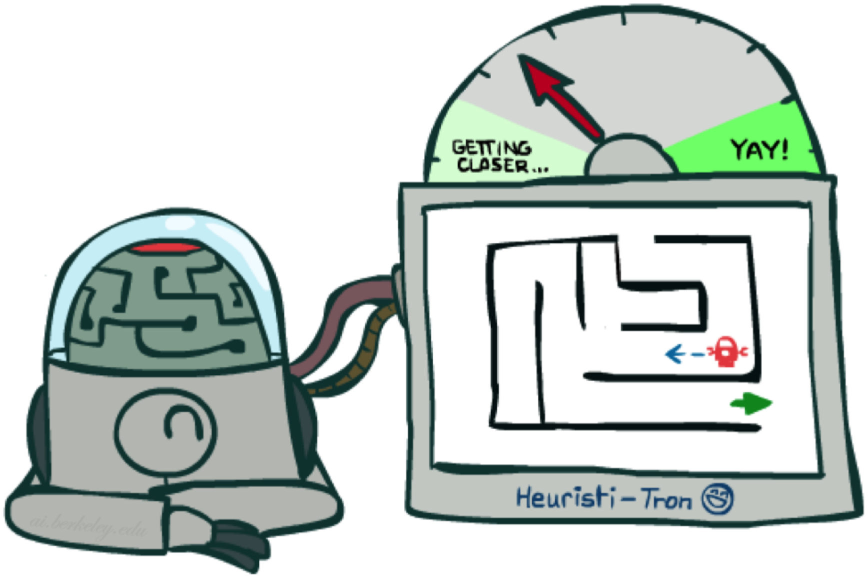
Admissible Heuristics



Idea: Admissibility



Inadmissible (pessimistic) heuristics break optimality by trapping good plans on the fringe



Admissible (optimistic) heuristics slow down bad plans but never outweigh true costs

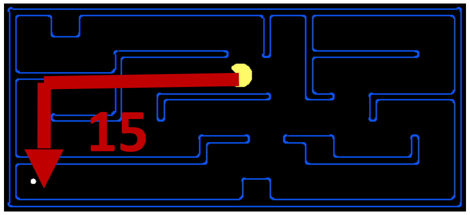
Admissible Heuristics

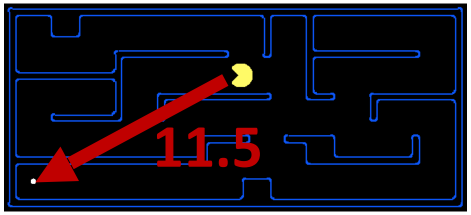
- A heuristic *h* is *admissible* (optimistic) iff:

$$0 \leq h(n) \leq h^*(n)$$

where $h^*(n)$ the true cost to a nearest goal

- Examples:

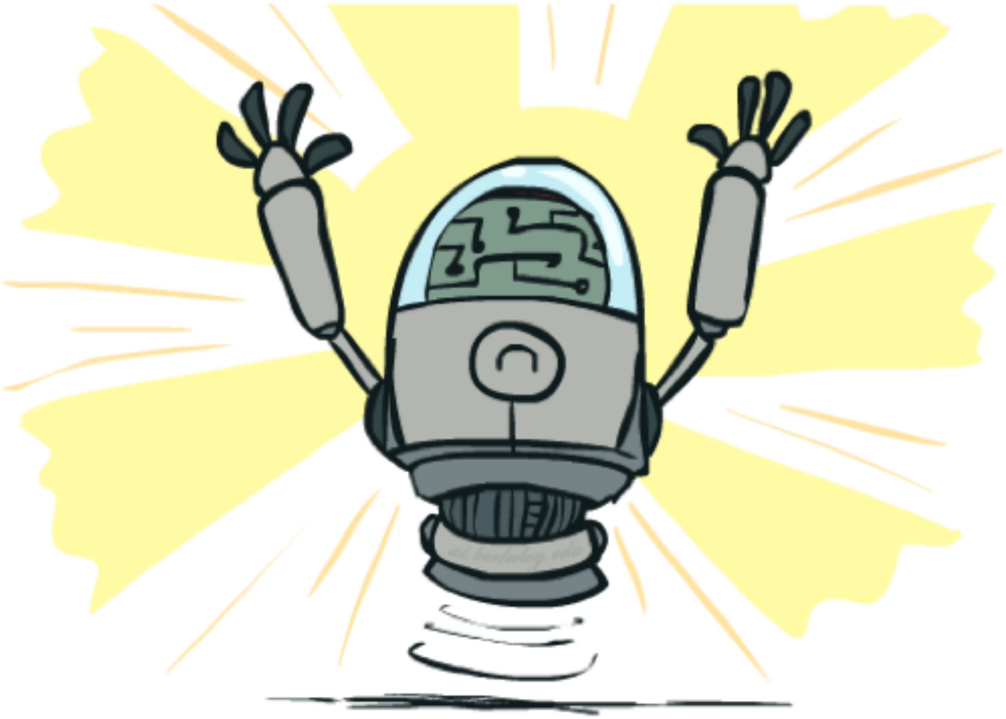




0.0

- Coming up with admissible heuristics is most of what's involved in using A* in practice.

Optimality of A* Tree Search



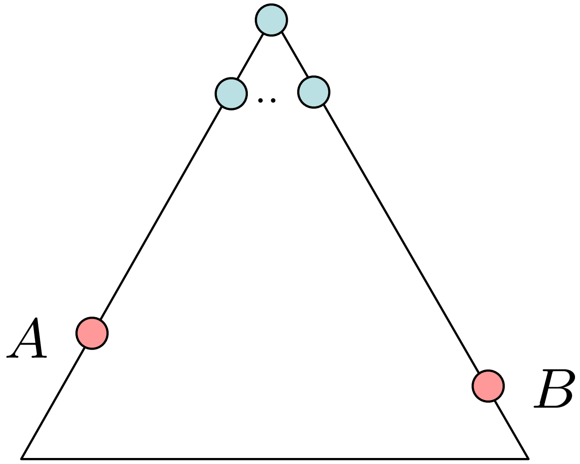
Optimality of A* Tree Search

Assume:

- A is an optimal goal node
- B is a suboptimal goal node
- h is admissible

Claim:

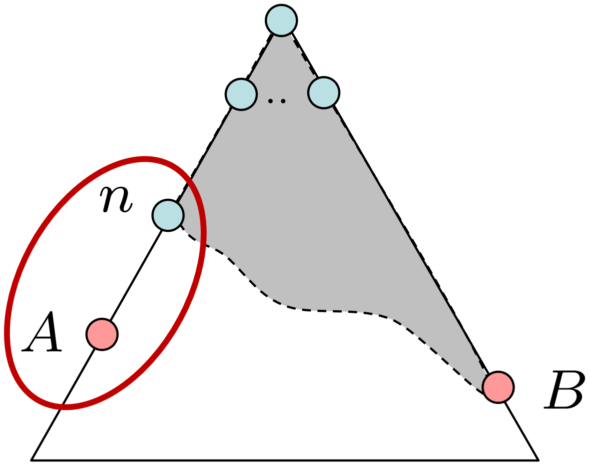
- A will exit the fringe before B



Optimality of A* Tree Search: Blocking

Proof:

- Imagine B is on the fringe
- Some ancestor *n* of A is on the fringe, too (maybe A!)
- Claim: *n* will be expanded before B
  1. f(n) is less or equal to f(A)

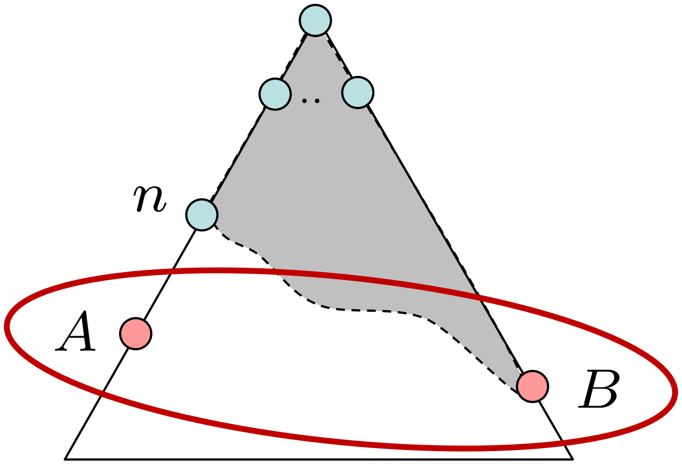


$f(n) = g(n) + h(n)$	Definition of f-cost
$f(n) \leq g(A)$	Admissibility of h
$g(A) = f(A)$	h = 0 at a goal

Optimality of A* Tree Search: Blocking

Proof:

- Imagine B is on the fringe
- Some ancestor *n* of A is on the fringe, too (maybe A!)
- Claim: *n* will be expanded before B
    1. f(n) is less or equal to f(A)
    2. f(A) is less than f(B)



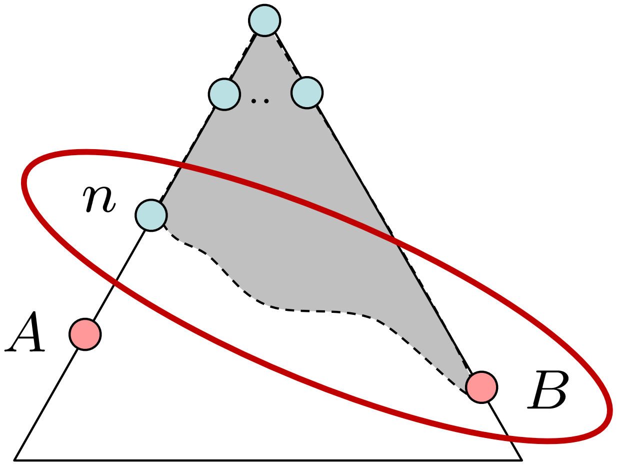
$$g(A) < g(B)$$ B is suboptimal

$$f(A) < f(B)$$ h = 0 at a goal

Optimality of A* Tree Search: Blocking

Proof:

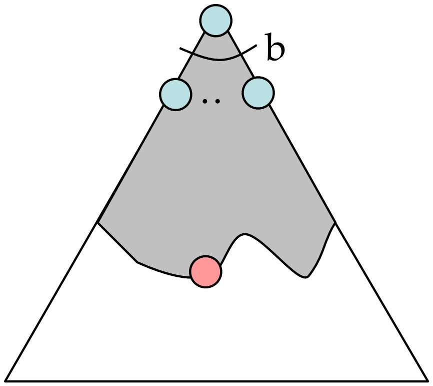
- Imagine B is on the fringe
- Some ancestor *n* of A is on the fringe, too (maybe A!)
- Claim: *n* will be expanded before B
  1. f(n) is less or equal to f(A)
  2. f(A) is less than f(B)
  3. *n* expands before B
- All ancestors of A expand before B
- A expands before B
- A* search is optimal



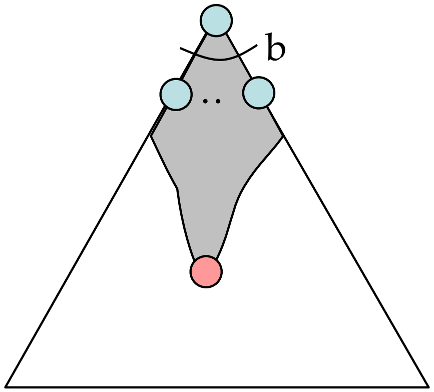
$$f(n) \leq f(A) < f(B)$$

Properties of A*

Uniform-Cost

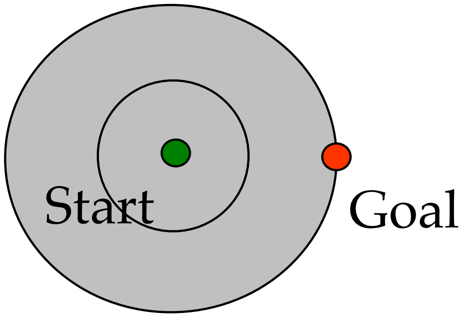


A*

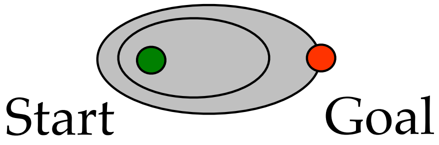


UCS vs A* Contours

- Uniform-cost expands equally in all "directions"



- A* expands mainly toward the goal, but does hedge its bets to ensure optimality



[Demo: contours UCS / greedy / A* empty (L3D1)]
[Demo: contours A* pacman small maze (L3D5)]

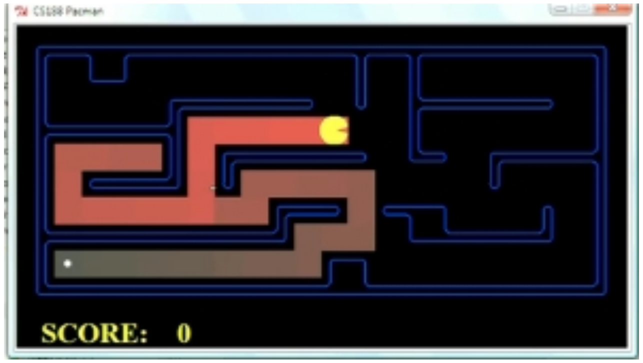
Video of Demo Contours (Empty) -- UCS

Video of Demo Contours (Empty) -- Greedy

Video of Demo Contours (Empty) – A*

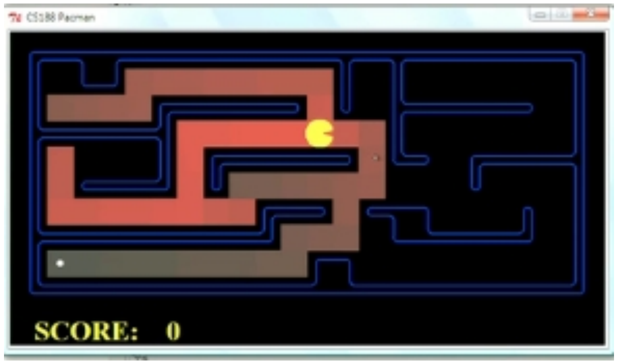
Demo Contours (Pacman Small Maze) – A*

Comparison



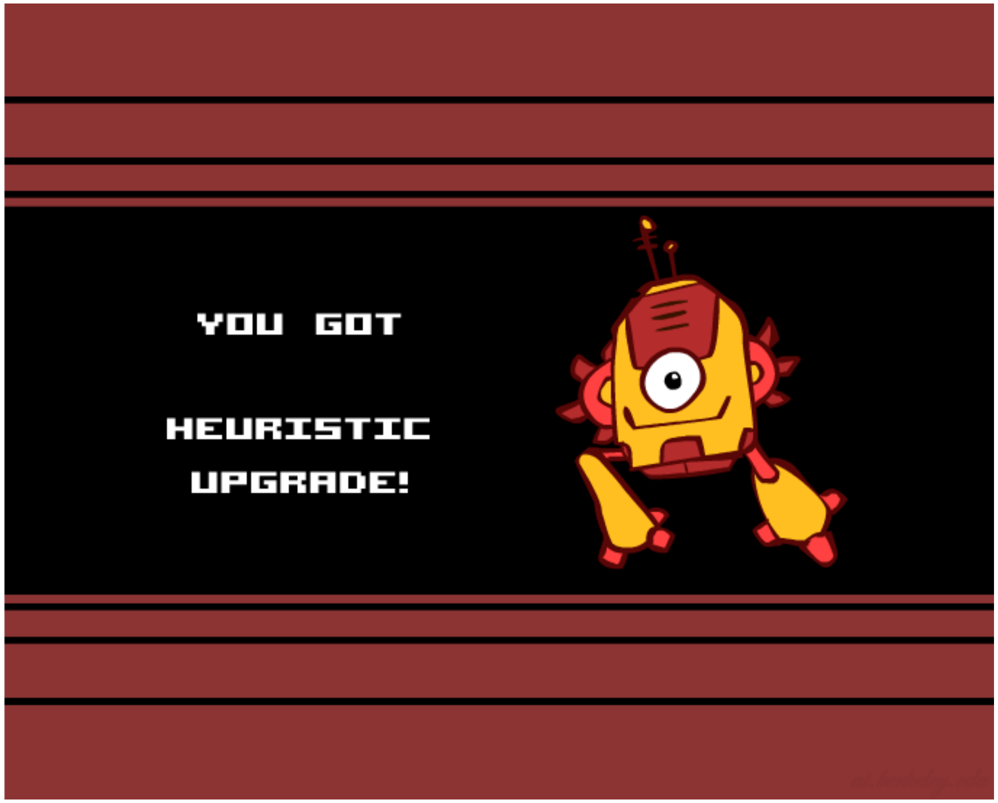
Greedy

Uniform Cost



A*

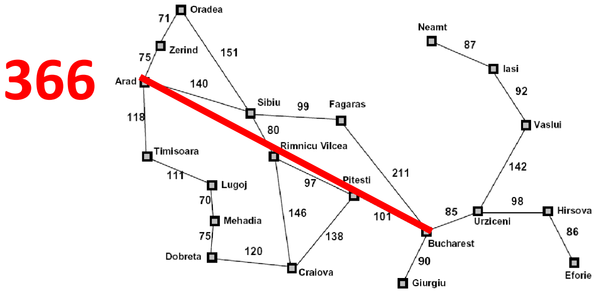
Creating Heuristics

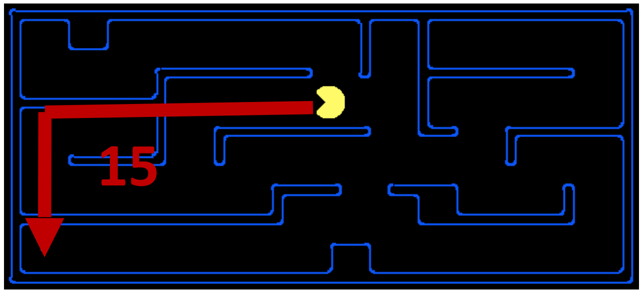


Creating Admissible Heuristics

- Most of the work in solving hard search problems optimally is in coming up with admissible heuristics

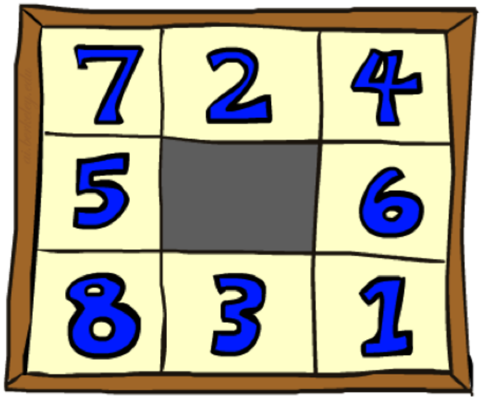
- Often, admissible heuristics are solutions to *relaxed problems*, where new actions are available



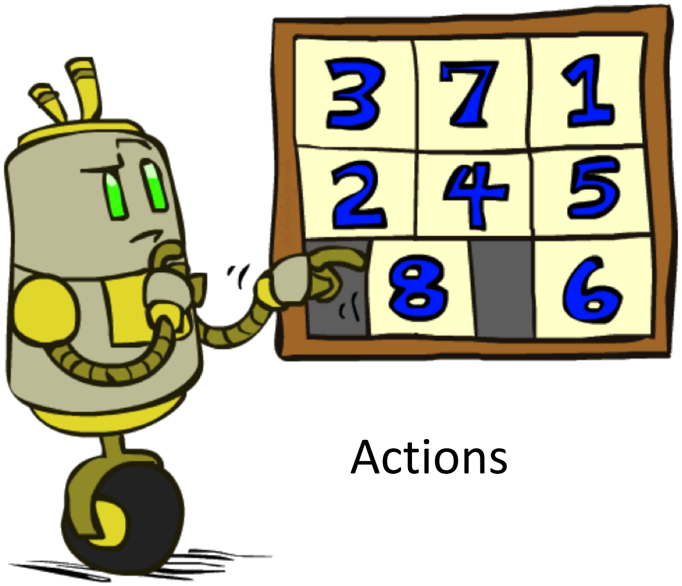


- Inadmissible heuristics are often useful too

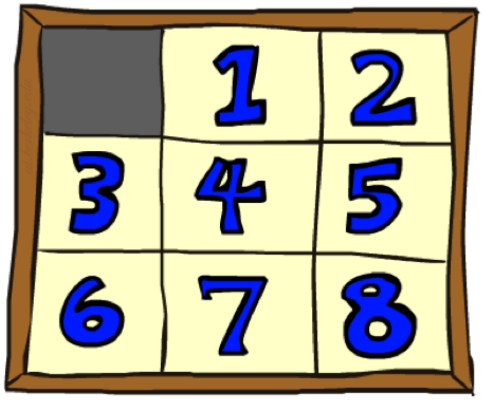
Example: 8 Puzzle



Start State



Actions



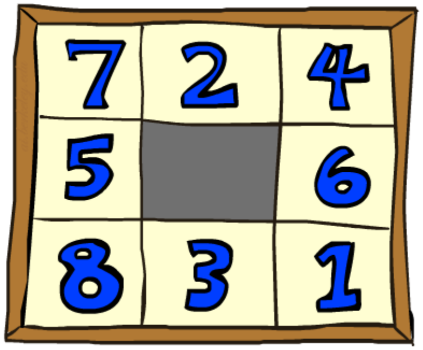
Goal State

- What are the states?
- How many states?
- What are the actions?
- How many successors from the start state?
- What should the costs be?

Admissible heuristics?

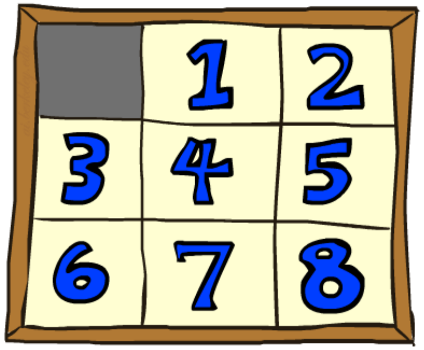
8 Puzzle I

- Heuristic: Number of tiles misplaced
- Why is it admissible?
- h(start) = 8
- This is a *relaxed-problem* heuristic



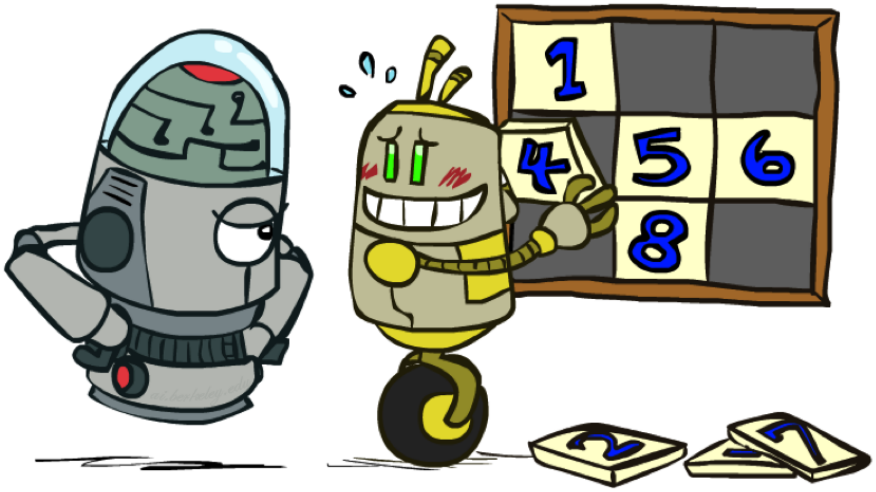
7	2	4
5		6
8	3	1

Start State



	1	2
3	4	5
6	7	8

Goal State

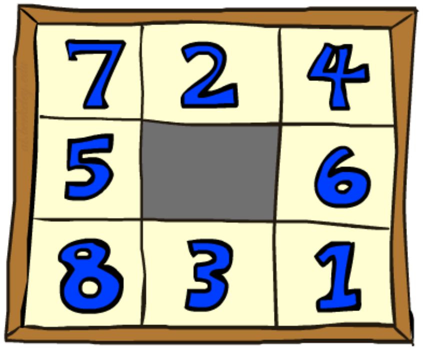


	Average nodes expanded when the optimal path has...		
	...4 steps	...8 steps	...12 steps
UCS	112	6,300	3.6×10^6
TILES	13	39	227

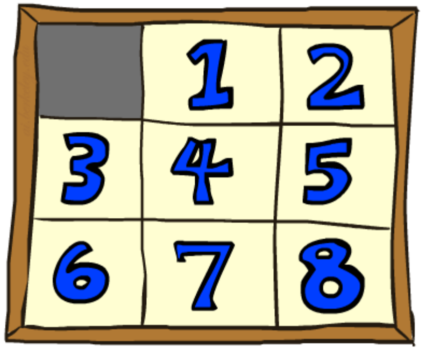
Statistics from Andrew Moore

8 Puzzle II

- What if we had an easier 8-puzzle where any tile could slide any direction at any time, ignoring other tiles?
- Total *Manhattan* distance
- Why is it admissible?
- h(start) = 3 + 1 + 2 + … = 18



Start State

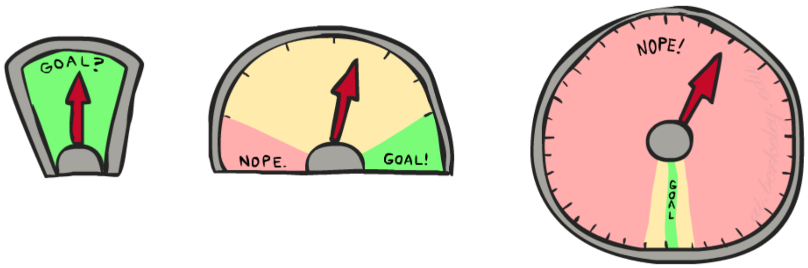


Goal State

	Average nodes expanded when the optimal path has…		
	…4 steps	…8 steps	…12 steps
TILES	13	39	227
MANHATTAN	12	25	73

8 Puzzle III

- How about using the *actual cost* as a heuristic?
  - Would it be admissible?
  - Would we save on nodes expanded?
  - What's wrong with it?



- With A*: a trade-off between quality of estimate and work per node
  - As heuristics get closer to the true cost, you will expand fewer nodes but usually do more work per node to compute the heuristic itself

Trivial Heuristics, Dominance

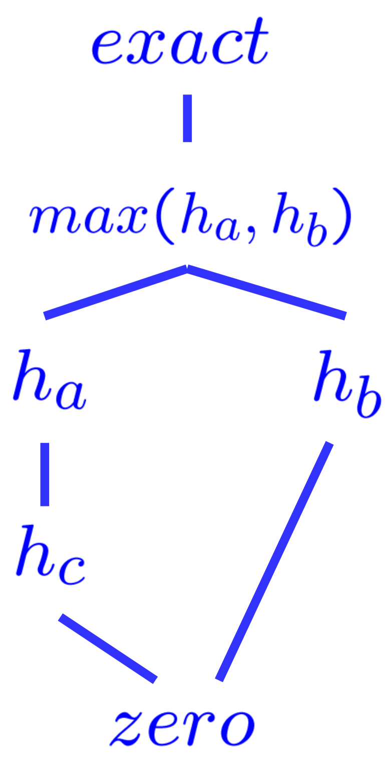
- Dominance: $h_a \geq h_c$ if

$$\forall n : h_a(n) \geq h_c(n)$$

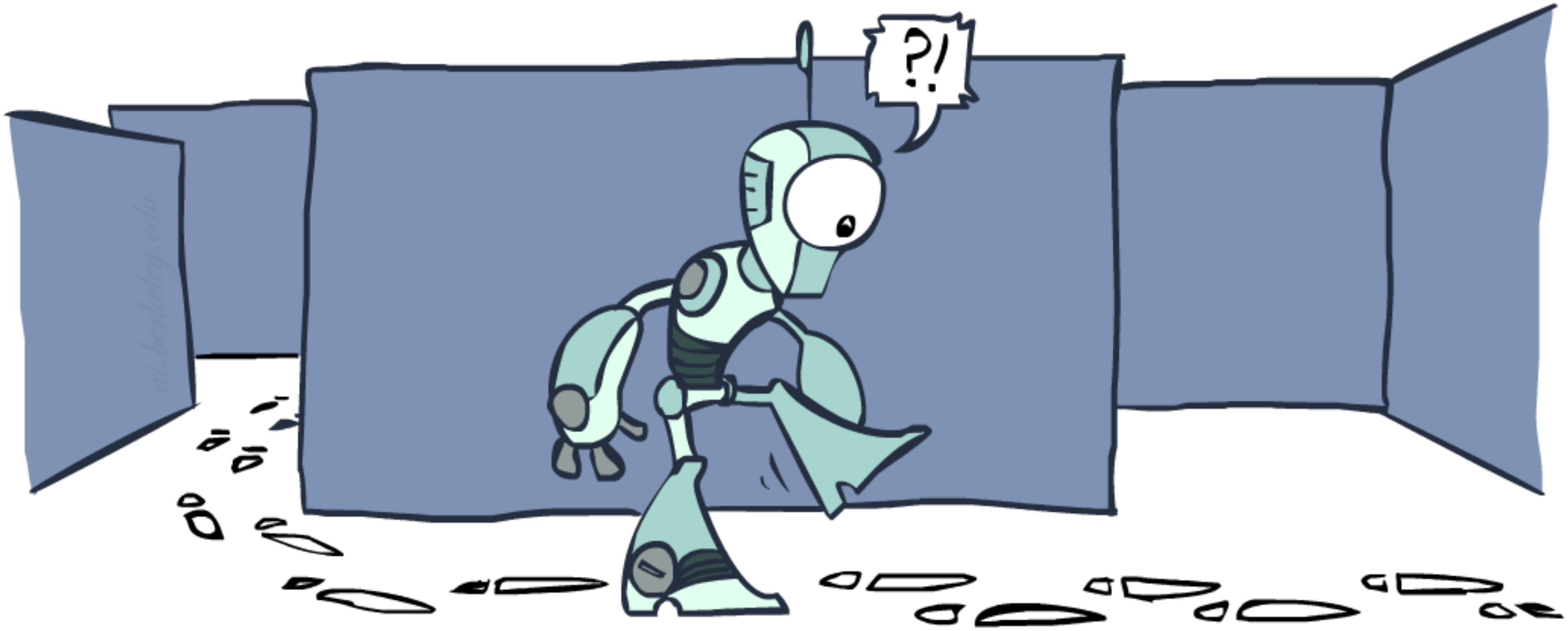
- Heuristics form a semi-lattice:
  - Max of admissible heuristics is admissible

$$h(n) = max(h_a(n), h_b(n))$$

- Trivial heuristics
  - Bottom of lattice is the zero heuristic (what does this give us?)
  - Top of lattice is the exact heuristic



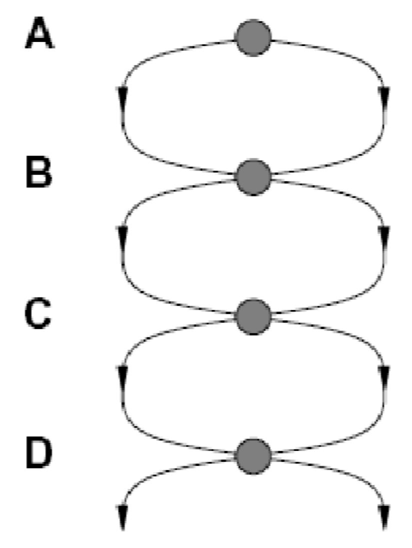
Graph Search



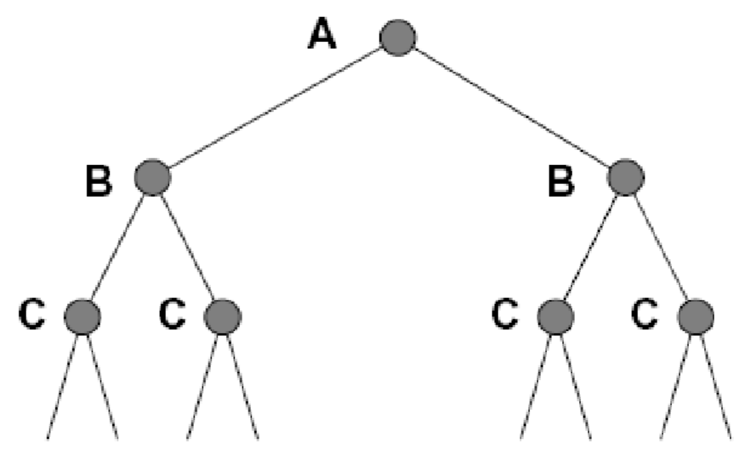
Tree Search: Extra Work!

- Failure to detect repeated states can cause exponentially more work.

State Graph

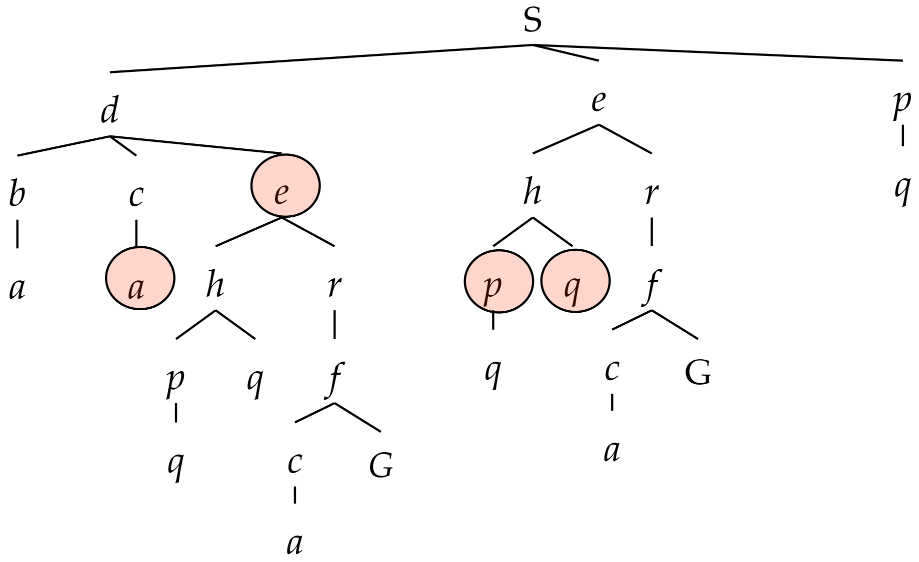


Search Tree



Graph Search

- In BFS, for example, we shouldn't bother expanding the circled nodes (why?)

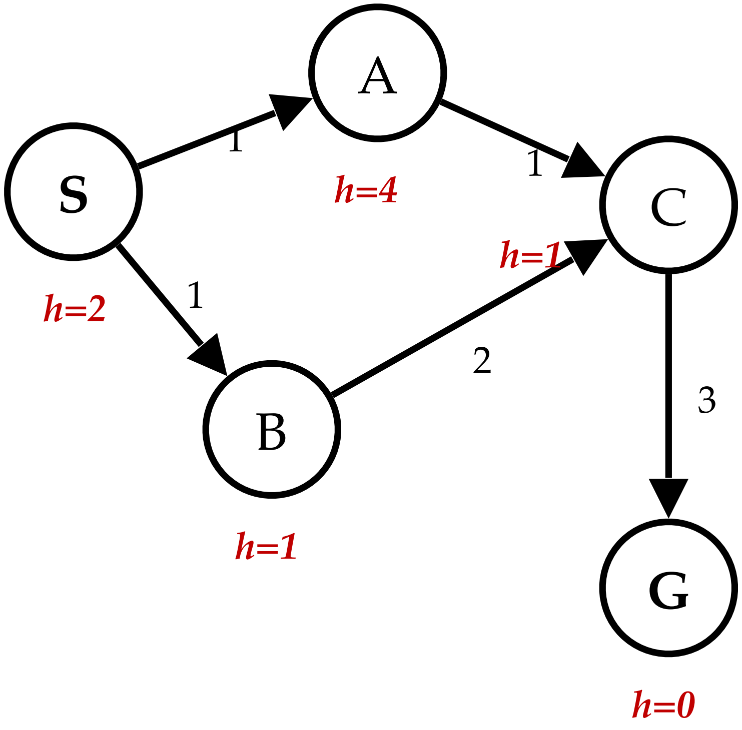


Graph Search

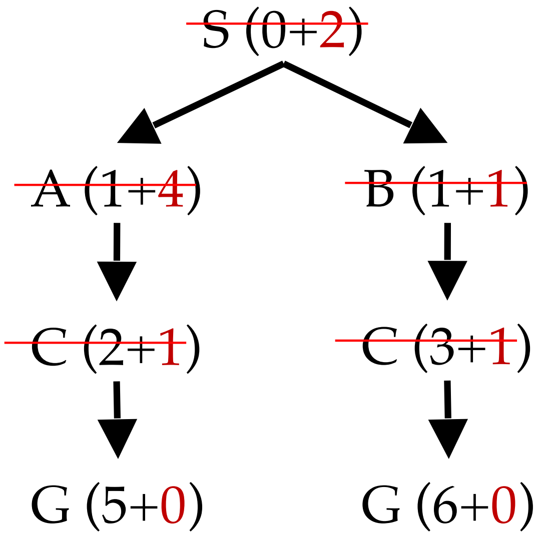
- Idea: never expand a state twice
- How to implement:
  - Tree search + set of expanded states (“closed set”)
  - Expand the search tree node-by-node, but…
  - Before expanding a node, check to make sure its state has never been expanded before
  - If not new, skip it, if new add to closed set
- Important: store the closed set as a set, not a list
- Can graph search wreck completeness? Why/why not?
- How about optimality?

A* Graph Search Gone Wrong?

State space graph



Search tree


~~S (0+2)~~

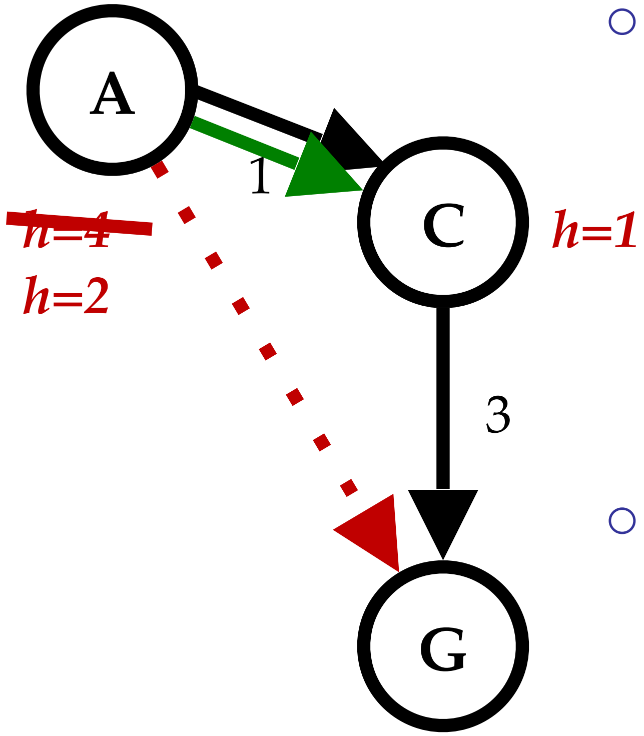
~~A (1+4)~~ ~~B (1+1)~~

~~C (2+1)~~ ~~C (3+1)~~

G (5+0) G (6+0)

Closed Set:S B C A

Consistency of Heuristics



- Main idea: estimated heuristic costs ≤ actual costs
  - Admissibility: heuristic cost ≤ actual cost to goal

    $h(v) \leq h^*(v)$ for all $v \in V$

    Underestimate the true cost to the goal!
  - Consistency: heuristic "arc" cost ≤ actual cost for each arc

    $h(v) - h(v) \leq d(u, v)$ for all $(u, v) \in E$

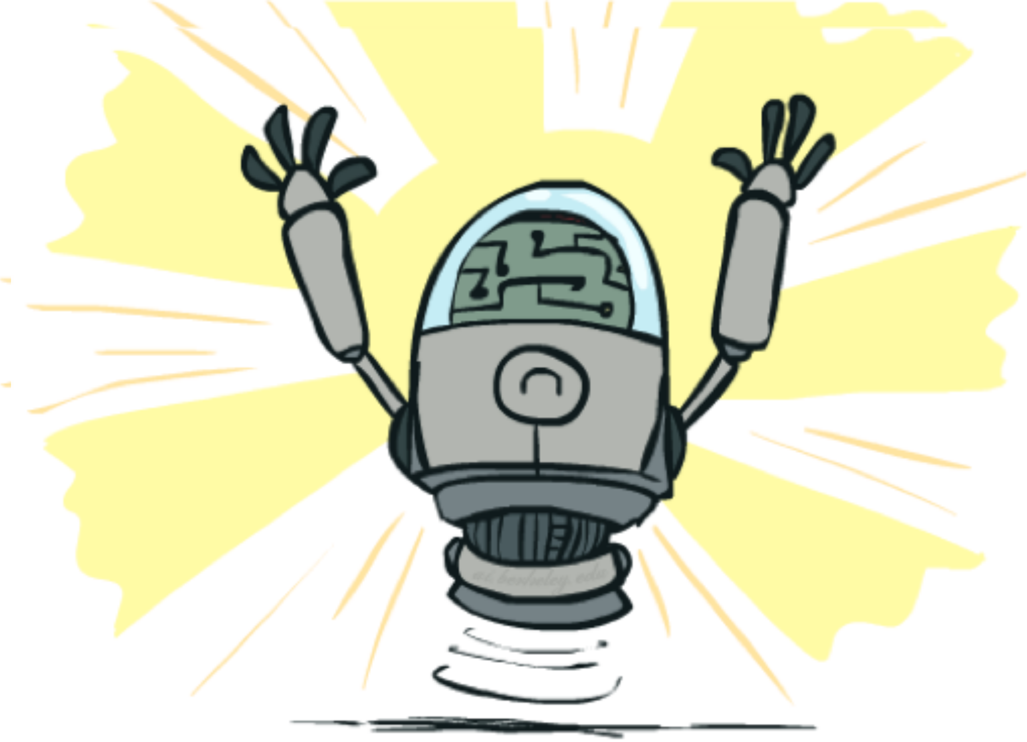
    Underestimate the weight of every edge!
- Consequences of consistency:
  - The f value along a path never decreases

    $h(A) \leq cost(A \text{ to } C) + h(C)$
  - A* graph search is optimal

Optimality of A* Search

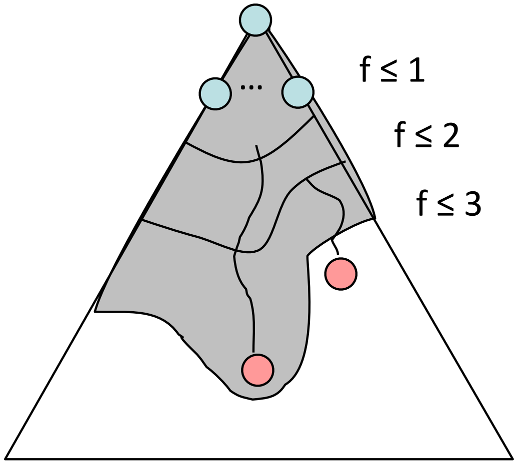
- With a admissible heuristic, Tree A* is optimal.
- With a consistent heuristic, Graph A* is optimal.
    - With h=0, the same proof shows that UCS is optimal.

Optimality of A* Graph Search



Optimality of A* Graph Search

- Sketch: consider what A* does with a consistent heuristic:
  - Fact 1: In tree search, A* expands nodes in increasing total f value (f-contours)
  - Fact 2: For every state s, nodes that reach s optimally are expanded before nodes that reach s suboptimally
  - Result: A* graph search is optimal



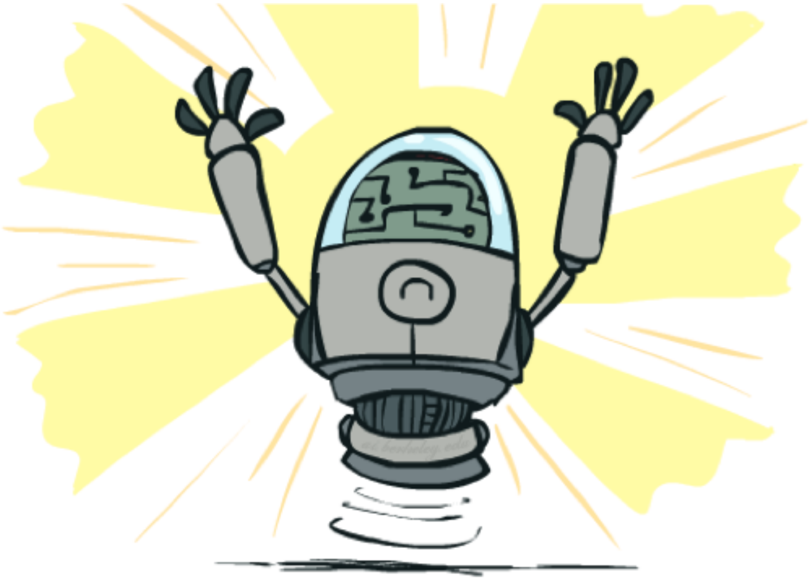
Optimality

- Tree search:
  - A* is optimal if heuristic is admissible
  - UCS is a special case (h = 0)

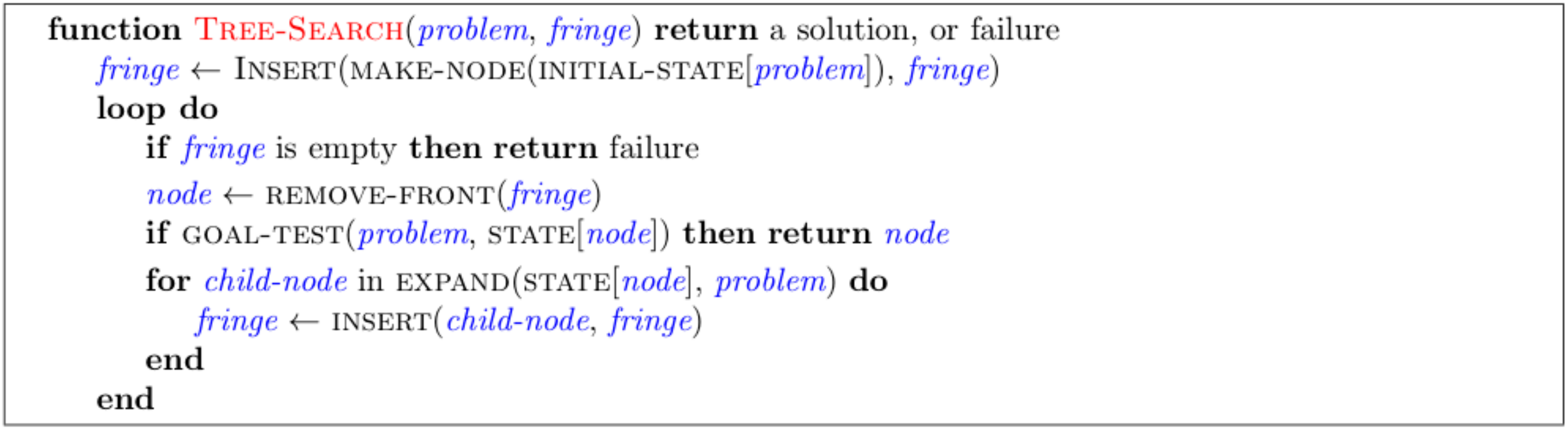
- Graph search:
  - A* optimal if heuristic is consistent
  - UCS optimal (h = 0 is consistent)

- Consistency implies admissibility

- In general, most natural admissible heuristics tend to be consistent, especially if it comes from a relaxed problem

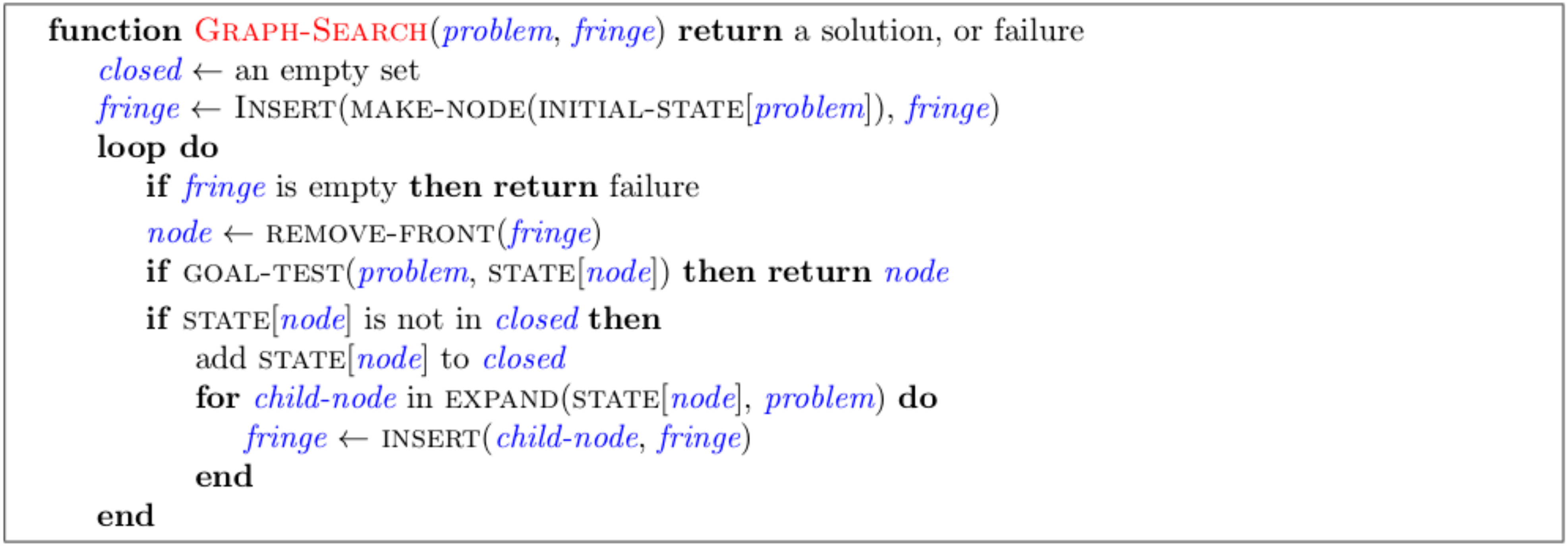


Tree Search Pseudo-Code



```
function TREE-SEARCH(problem, fringe) return a solution, or failure
    fringe ← INSERT(MAKE-NODE(INITIAL-STATE[problem]), fringe)
    loop do
        if fringe is empty then return failure
        node ← REMOVE-FRONT(fringe)
        if GOAL-TEST(problem, STATE[node]) then return node
        for child-node in EXPAND(STATE[node], problem) do
            fringe ← INSERT(child-node, fringe)
        end
    end
```

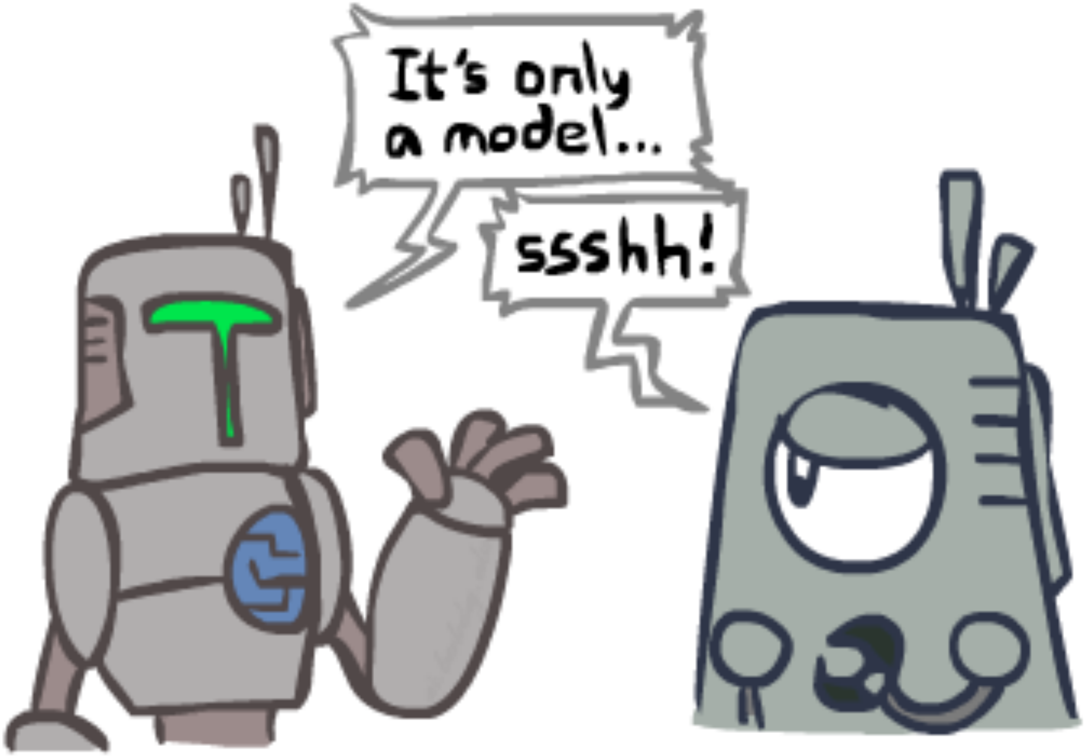
Graph Search Pseudo-Code



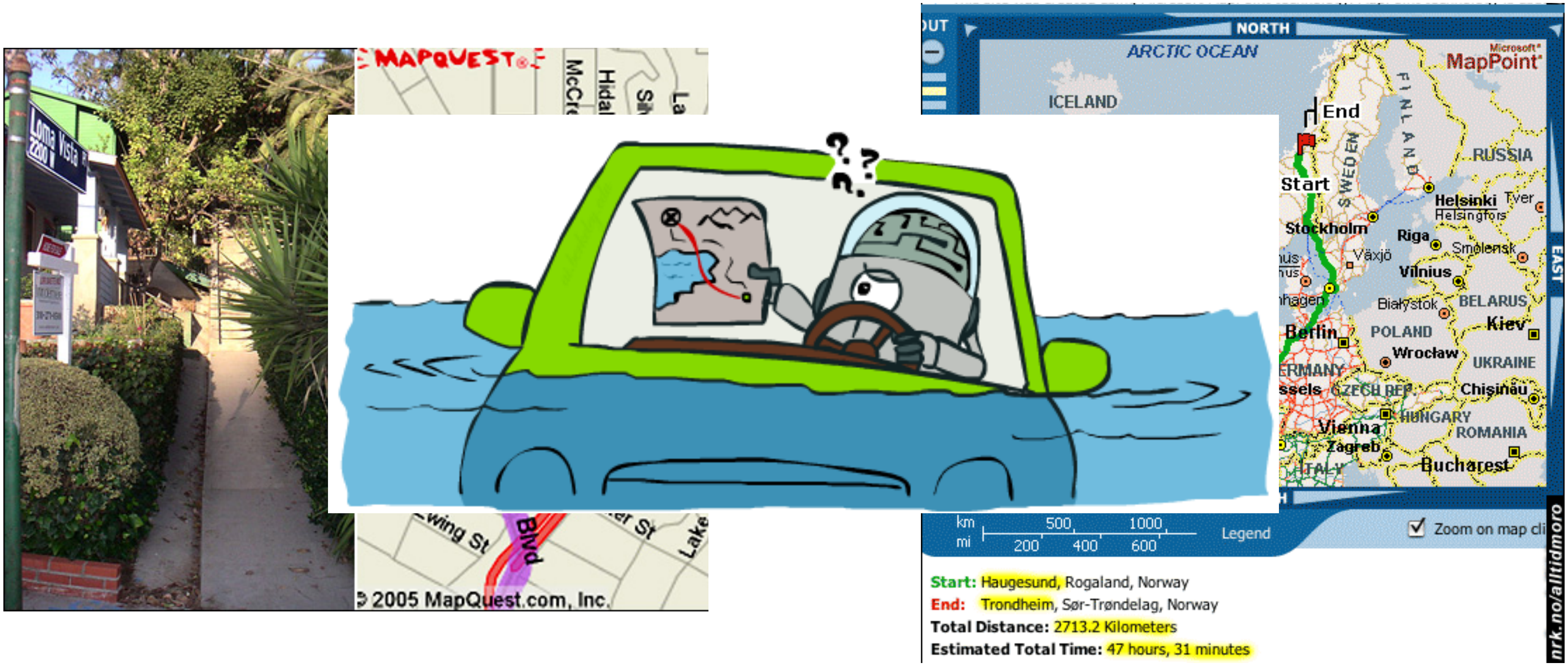
```
function GRAPH-SEARCH(problem, fringe) return a solution, or failure
    closed ← an empty set
    fringe ← INSERT(MAKE-NODE(INITIAL-STATE[problem]), fringe)
    loop do
        if fringe is empty then return failure
        node ← REMOVE-FRONT(fringe)
        if GOAL-TEST(problem, STATE[node]) then return node
        if STATE[node] is not in closed then
            add STATE[node] to closed
            for child-node in EXPAND(STATE[node], problem) do
                fringe ← INSERT(child-node, fringe)
            end
    end
```

Search and Models

- Search operates over models of the world
  - The agent doesn't actually try all the plans out in the real world!
  - Planning is all "in simulation"
  - Your search is only as good as your models…



Search Gone Wrong?



Start: Haugesund, Rogaland, Norway
End: Trondheim, Sør-Trøndelag, Norway
Total Distance: 2713.2 Kilometers
Estimated Total Time: 47 hours, 31 minutes