1 Readings

To date we have covered the following material in Benenti, Casati, and Strini:

Ch.2

Ch. 3.1 - 3.4

2 EPR pairs and information transfer

Nature is consistent with quantum mechanics and not with local realism, confirming that for the wavefunction

$$\psi = \alpha|0\rangle + \beta|1\rangle$$

nothing can be known about the coefficients α, β until a measurement is made.

Entangled pairs of qubits such as

$$|\psi_{ab}\rangle = \frac{1}{\sqrt{2}}\left(|0_a 1_b\rangle + |1_a 0_b\rangle\right)$$

can be used to *facilitate* sharing or transmission of information, but *not* to transmit information from A to B directly. I.e., there is no superluminal transfer of information happening in an entangled state. Why not? Because Alice has no control over the result of her measurement and consequently she cannot control what Bob measures either.

Many names have been given to describe the effects of entanglement:

"quantum non-locality"

"spooky action-at-a-distance" (Einstein)

"passion-at-a-distance" (A. Shimony)

I can add "belonging-at-a-distance" to these descriptors of the relation between the measurements made on qubits a and b.

2.1 Tensor product of operators

Suppose $|v\rangle$ and $|w\rangle$ are unentangled states on $\mathscr{C}^m$ and $\mathscr{C}^n$, respectively. The state of the combined system is $|v\rangle \otimes |w\rangle$ on $\mathscr{C}^{mn}$. If the unitary operator A is applied to the first subsystem, and B to the second subsystem, the combined state becomes $A|v\rangle \otimes B|w\rangle$.

In general, the two subsystems will be entangled with each other, so the combined state is not a tensor-product state. We can still apply A to the first subsystem and B to the second subsystem. This gives the operator $A \otimes B$ on the combined system, defined on entangled states by linearly extending its action on unentangled states.

(For example, $(A\otimes B)(|0\rangle \otimes |0\rangle) = A|0\rangle \otimes B|0\rangle$. $(A\otimes B)(|1\rangle \otimes |1\rangle) = A|1\rangle \otimes B|1\rangle$. Therefore, we define $(A\otimes B)(\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle)$ to be $\frac{1}{\sqrt{2}}(A\otimes B)|00\rangle + \frac{1}{\sqrt{2}}(A\otimes B)|11\rangle = \frac{1}{\sqrt{2}}\left(A|0\rangle \otimes B|0\rangle + A|1\rangle \otimes B|1\rangle\right)$.)

Let $|e_1\rangle, \ldots, |e_m\rangle$ be a basis for the first subsystem, and write $A = \sum_{i,j=1}^m a_{ij}|e_i\rangle\langle e_j|$ (the i,jth element of A is a_{ij}). Let $|f_1\rangle, \ldots, |f_n\rangle$ be a basis for the second subsystem, and write $B = \sum_{k,l=1}^n b_{kl}|f_k\rangle\langle f_l|$. Then a basis for the combined system is $|e_i\rangle \otimes |f_j\rangle$, for $i = 1, \ldots, m$ and $j = 1, \ldots, n$. The operator $A\otimes B$ is

$$\begin{aligned} A\otimes B &= \left(\sum_{ij} a_{ij}|e_i\rangle\langle e_j|\right) \otimes \left(\sum_{kl} b_{kl}|f_k\rangle\langle f_l|\right) \\ &= \sum_{ijkl} a_{ij}b_{kl}|e_i\rangle\langle e_j| \otimes |f_k\rangle\langle f_l| \\ &= \sum_{ijkl} a_{ij}b_{kl}(|e_i\rangle \otimes |f_k\rangle)(\langle e_j| \otimes \langle f_l|) \ . \end{aligned}$$

Therefore the $(i,k),(j,l)$th element of $A\otimes B$ is $a_{ij}b_{kl}$. If we order the basis $|e_i\rangle \otimes |f_j\rangle$ lexicographically (i.e. first according to the index i, then according to the index j), then the matrix for $A\otimes B$ is

$$\begin{pmatrix} a_{11}B & a_{12}B & \cdots \\ a_{21}B & a_{22}B & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} \ ;$$

i.e. in the i,jth subblock, we multiply a_{ij} by the matrix for B.

For practice with these tensor operators, see the examples worked out in the previous lecture, specifically in the section calculating quantum analogs of the classical correlation functions of different measurements for the Bell inequality. For example, work through the evaluation of

$$\begin{aligned} \langle A\otimes B'\rangle &= \langle \psi^-|A\otimes B'|\psi^-\rangle \\ |\psi_{ab}\rangle &= \frac{1}{\sqrt{2}}\left(|0_a 1_b\rangle + |1_a 0_b\rangle\right) \end{aligned}$$

with $A = \sigma_z^a$ and $B' = \cos\phi\,\sigma_z^b - \sin\phi\,\sigma_x^b$.

3 Example of more efficient information processing by use of shared entanglement

Consider the following communication protocol in the classical world: Alice (A) and Bob (B) are two parties who share a common string S. They receive independent, random bits X_A, X_B, and try to output bits a, b respectively, such that $X_A \wedge X_B = a \oplus b$. (The notation $x \wedge y$ takes the AND of two binary variables x and y, i.e., is one if $x = y = 1$ and zero otherwise. $x \oplus y \equiv x + y \bmod 2$, the XOR.)

In the quantum mechanical analogue of this protocol, A and B share the EPR pair $|\Psi^-\rangle$. As before, they receive bits X_A, X_B, and try to output bits a, b respectively, such that $X_A \wedge X_B = a \oplus b$.

However, Alice and Bob's best protocol for the classical game, as you will prove in the homework, is to output $a = 0$ and $b = 0$, respectively. Then $a \oplus b = 0$, so as long as the inputs $(X_A, X_B) \neq (1,1)$, they are successful: $a \oplus b = 0 = X_A \wedge X_B$. If $X_A = X_B = 1$, then they fail. Therefore they are successful with probability exactly $3/4$.

We will show that the quantum mechanical system can do better. Specifically, if Alice and Bob share an EPR pair, we will describe a protocol for which the probability $\Pr\{X_A \wedge X_B = a \oplus b\}$ is greater than 3/4.

We can setup the following protocol:

- if $X_A = 0$, then Alice measures in the standard basis, and outputs the result.
- if $X_A = 1$, then Alice rotates by $\pi/8$, then measures, and outputs the result.
- if $X_B = 0$, then Bob measures in the standard basis, and outputs the complement of the result.
- if $X_B = 1$, then Bob rotates by $-\pi/8$, then measures, and outputs the complement of the result.

So we need to calculate $\Pr\{a \oplus b \neq X_A \wedge X_B\}$ for each of the four possible cases:

$$\Pr\{a \oplus b \neq X_A \wedge X_B\} = \sum_{X_A, X_B} \tfrac{1}{4} \Pr\{a \oplus b \neq X_A \wedge X_B \big| X_A, X_B\}$$

First we note that if measurement in the standard basis yields $|0\rangle$ with probability 1, then if a state is rotated by θ, measurement will yield $|0\rangle$ with probability $\cos^2(\theta)$. [Recall that in general, rotation of a state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ by angle θ in the two-dimensional state space gives the rotated state $|\psi'\rangle = \alpha'|0\rangle + \beta'|1\rangle$, where

$$\begin{pmatrix} \alpha' \\ \beta' \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}. \tag{1}$$

Hence the probability of measuring a 0 for the rotated state is given by $\alpha^2 \cos^2(\theta)$, etc.]

Now we claim

$$\begin{aligned}
\Pr\{a \oplus b \neq X_A \wedge X_B \big| X_A = 0, X_B = 0\} &= 0 \\
\Pr\{a \oplus b \neq X_A \wedge X_B \big| X_A = 0, X_B = 1\} &= \sin^2(\pi/8) \\
\Pr\{a \oplus b \neq X_A \wedge X_B \big| X_A = 1, X_B = 0\} &= \sin^2(\pi/8) \\
\Pr\{a \oplus b \neq X_A \wedge X_B \big| X_A = 1, X_B = 1\} &= \cos^2(\pi/4) = 1/2 \ .
\end{aligned}$$

Indeed, for the first case, $X_A = X_B = 0$ (so $X_A \wedge X_B = 0$), Alice and Bob each measure in the computational basis, without any rotation. If Alice measures $a = 0$, then Bob's measurement is the opposite, and Bob outputs the complement, $b = 0$. Therefore $a \oplus b = 0 = X_A \wedge X_B$, a success. Similarly if Alice measures $a = 1$, they are always successful.

In the second case, $X_A = 0$, $X_B = 1$ ($X_A \wedge X_B = 0$). If Alice measures $a = 0$, then the new state of the system is $|01\rangle$; Bob's qubit is in the state $|1\rangle$. In the rotated basis, Bob measures a 1 (and outputs its complement, 0) with probability $\cos^2(\pi/8)$. The probability of *failure* is therefore $1 - \cos^2(\pi/8) = \sin^2(\pi/8)$. Similarly if Alice measures $a = 1$. The third case, $X_A = 1$, $X_B = 0$ is symmetrical and gives the same result.

In the final case, $X_A = X_B = 1$ (so $X_A \wedge X_B = 1$), Alice and Bob are measuring in bases rotated 45 degrees from each other. If Alice measures $a = 0$, then Bob measures a 1 and outputs a 0 with probability $\cos^2(\pi/4)$. This gives $a \oplus b = 0 \neq X_A \wedge X_B$, i.e., a failure. Similarly if Alice measures $a = 1$. So the probability of failure is now $\cos^2(\pi/4) = 1/2$.

Averaging over the four cases, we find

$$\begin{aligned}
\Pr\left\{a \oplus b \neq X_A \wedge X_B\right\} &= 1/4\left(2\sin^2(\pi/8) + 1/2\right) \\
&= 1/4\,(1 - \cos(2 * \pi/8) + 1/2) \\
&= 1/4\left(3/2 - \sqrt{2}/2\right) \\
&\approx 1/8\,(3 - 1.4) \\
&= 1.6/8 = .2 \quad .
\end{aligned}$$

The probability of success with this protocal is therefore around .8, better than any protocol could achieve with a classical model.