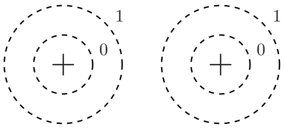
Two Qubit Entanglement and Bell Inequalities.

1 Two qubits:

Now let us examine a system of two qubits. Consider the two electrons in two hydrogen atoms, each regarded as a 2-state quantum system:



Since each electron can be in either of the ground or excited state, classically the two electrons are in one of four states – 00, 01, 10, or 11 – and represent 2 bits of classical information. By the superposition principle, the quantum state of the two electrons can be any linear combination of these four classical states:

$$|\psi\rangle = \alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{10}|10\rangle + \alpha_{11}|11\rangle \quad ,$$

where $\alpha_{ij} \in \mathscr{C}$, $\sum_{ij} |\alpha_{ij}|^2 = 1$. Again, this is just Dirac notation for the unit vector in $\mathscr{C}^4$:

$$\begin{pmatrix} \alpha_{00} \\ \alpha_{01} \\ \alpha_{10} \\ \alpha_{11} \end{pmatrix}$$

Measurement:

Measuring $|\psi\rangle$ now reveals two bits of information. The probability that the outcome of the measurement is the two bit string $x \in \{0,1\}^2$ is $|\alpha_x|^2$. Moreover, following the measurement the state of the two qubits is $|x\rangle$. i.e. if the first bit of x is j and the second bit k, then following the measurement, the state of the first qubit is $|j\rangle$ and the state of the second is $|k\rangle$.

An interesting question comes up here: what if we measure just the first qubit? What is the probability that the outcome is 0? This is simple. It is exactly the same as it would have been if we had measured both qubits: $\Pr\{1\text{st bit } = 0\} = \Pr\{00\} + \Pr\{01\} = |\alpha_{00}|^2 + |\alpha_{01}|^2$. Ok, but how does this partial measurement disturb the state of the system?

The answer is obtained by an elegant generalization of our previous rule for obtaining the new state after a measurement. The new superposition is obtained by crossing out all those terms of $|\psi\rangle$ that are inconstent with the outcome of the measurement (i.e. those whose first bit is 1). Of course, the sum of the squared amplitudes is no longer 1, so we must renormalize to obtain a unit vector:

$$|\phi\rangle_{new} = \frac{\alpha_{00}|00\rangle + \alpha_{01}|01\rangle}{\sqrt{|\alpha_{00}|^2 + |\alpha_{01}|^2}}$$

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Entanglement

Suppose the first qubit is in the state $3/5|0\rangle + 4/5|1\rangle$ and the second qubit is in the state $1/\sqrt{2}|0\rangle - 1/\sqrt{2}|1\rangle$, then the joint state of the two qubits is $(3/5|0\rangle + 4/5|1\rangle)(1/\sqrt{2}|0\rangle - 1/\sqrt{2}|1\rangle) = 3/5\sqrt{2}|00\rangle - 3/5\sqrt{2}|01\rangle + 4/5\sqrt{2}|10\rangle - 4/5\sqrt{2}|11\rangle$

But there are states such as $|\Phi^+\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)$ which cannot be decomposed in this way as a state of the first qubit and that of the second qubit. Can you see why? Such a state is called an entangled state.

If the first (resp. second) qubit of $|\Phi^+\rangle$ is measured then the outcome is 0 with probability $1/2$ and 1 with probability $1/2$. However if the outcome is 0, then a measurement of the second qubit results in 0 with certainty. Furthermore this is true even if both qubits are measured in a rotated basis $|v\rangle, |v^\perp\rangle$, where $|v\rangle = \alpha|0\rangle + \beta|1\rangle$ and $|v^\perp\rangle = -\beta|0\rangle + \alpha|1\rangle$.

Claim: $|\Phi^+\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)$
$= \frac{1}{\sqrt{2}}\left(|vv\rangle + |v^\perp v^\perp\rangle\right)$.

Proof: Then $\frac{1}{\sqrt{2}}(|vv\rangle + |v^\perp v^\perp\rangle)$
$= \frac{1}{\sqrt{2}}(\alpha^2|00\rangle + \alpha\beta|01\rangle + \alpha\beta|10\rangle + \beta^2|11\rangle) + \frac{1}{\sqrt{2}}(\beta^2|00\rangle - \alpha\beta|01\rangle - \alpha\beta|10\rangle + \alpha^2|11\rangle)$
$= \frac{1}{\sqrt{2}}(\alpha^2 + \beta^2)(|00\rangle + |11\rangle)$
$= \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$

1.1 Two Qubit Gates

Let us now consider how a system of two qubits evolves in time. Recall that the third axiom of quantum physics states that the evolution of a quantum system is necessarily unitary. Intuitively, a unitary transformation is a rigid body rotation (or reflection) of the Hilbert space, thus resulting in a transformation of the state vector that doesn't change its length.

Let us consider what this means for the evolution of a two qubit system. A unitary transformation on the Hilbert space $\mathscr{C}^4$ is specified by a 4x4 matrix U that satisfies the condition $UU^\dagger = U^\dagger U = I$. The four columns of U specify the four orthonormal vectors $|v_{00}\rangle$, $|v_{01}\rangle$, $|v_{10}\rangle$ and $|v_{11}\rangle$ that the basis states $|00\rangle$, $|01\rangle$, $|10\rangle$ and $|11\rangle$ are mapped to by U.

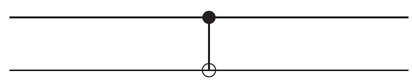
A very basic two qubit gate is the controlled-not gate or the CNOT:

- Controlled Not (CNOT).

$$\text{CNOT} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

  The first bit of a CNOT gate is the "control bit;" the second is the "target bit." The control bit never changes, while the target bit flips if and only if the control bit is 1.

  The CNOT gate is usually drawn as follows, with the control bit on top and the target bit on the bottom:

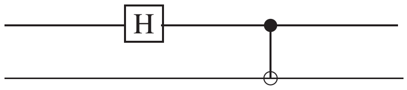


Though the CNOT gate looks very simple, any unitary transformation on two qubits can be closely approximated by a sequence of CNOT gates and single qubit gates. This brings us to an important point. What happens to the quantum state of two qubits when we apply a single qubit gate to one of them,

say the first? Let's do an example. Suppose we apply a Hadamard gate to the superposition: $|\psi\rangle = 1/2|00\rangle - i/\sqrt{2}|01\rangle + 1/\sqrt{2}|11\rangle$. Then this maps the first qubit as follows: $|0\rangle \rightarrow 1/\sqrt{2}|0\rangle + 1/\sqrt{2}|1\rangle$, and
$|1\rangle \rightarrow 1/\sqrt{2}|0\rangle - 1/\sqrt{2}|1\rangle$.
So $|\psi\rangle \rightarrow 1/2\sqrt{2}|00\rangle + 1/2\sqrt{2}|01\rangle - i/2|00\rangle + i/2|01\rangle + 1/2|10\rangle - 1/2|11\rangle$
$= (1/2\sqrt{2} - i/2)|00\rangle + (1/2\sqrt{2} + i/2)|01\rangle + 1/2|10\rangle - 1/2|11\rangle$.

Bell states:

We can generate the Bell states $|\Phi^+\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)$ with the following simple qauntum circuit consisting of a Hadamard and CNOT gate:



The first qubit is passed through a Hadamard gate and then both qubits are entangled by a CNOT gate.

If the input to the system is $|0\rangle \otimes |0\rangle$, then the Hadamard gate changes the state to

$$\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes |0\rangle = \frac{1}{sqrt2}|00\rangle + \frac{1}{\sqrt{2}}|10\rangle \quad ,$$

and after the CNOT gate the state becomes $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$, the Bell state $|\Phi^+\rangle$.

The state $|\Phi^+\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)$ is one of four Bell basis states:

$$\begin{aligned} |\Phi^\pm\rangle &= \frac{1}{\sqrt{2}}\left(|00\rangle \pm |11\rangle\right) \\ |\Psi^\pm\rangle &= \frac{1}{\sqrt{2}}\left(|01\rangle \pm |10\rangle\right) \quad . \end{aligned}$$

These are maximally entangled states on two qubits. Show how to generate all these states by a simple quantum circuit, and verify that the four Bell states form an orthonormal basis.

1.2 EPR Paradox:

Everyone has heard Einstein's famous quote "God does not play dice". It is lifted from Einstein's 1926 letter to Max Born where he expressed his dissatisfaction with quantum physics by writing: "Quantum mechanics is certainly imposing. But an inner voice tells me that it is not yet the real thing. The theory says a lot, but does not really bring us any closer to the secret of the Old One. I, at any rate, am convinced that He does not throw dice." Even to the end of his life he held on to the view that quantum physics is just an incomplete theory and that some day we would learn a more complete and satisfactory theory that describes nature. For example, consider coin-flipping. We can model coin-flipping as a random process giving heads 50% of the time, and tails 50% of the time. This model is perfectly predictive, but incomplete. If we knew the initial conditions of the coin with perfect accuracy (position, momentum), then we could solve Newton's equations to determine the eventual outcome of the coin flip with certainty.

Einstein sharpened this line of reasoning in a paper he wrote with Podolsky and Rosen in 1935, where they introduced the famous Bell states. Recall that for Bell state $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$, when you measure first qubit, the second qubit is determined. However, if two qubits are far apart, then the second qubit must have had a determined state in some time interval before measurement, since the speed of light is finite. Moreover this holds in any basis. This appears analogous to the coin flipping example. EPR therefore suggested that there is a more complete theory where "God does not throw dice." Until his death in 1955, Einstein tried to formulate a more complete "local hidden variable theory" that would describe the predictions of quantum mechanics, but without resorting to probabilistic outcomes. But in 1964, almost three decades after the EPR paper, John Bell showed that properties of Bell (EPR) states were not merely fodder for a philosophical discussion, but had verifiable consequences: local hidden variables are not the answer.

How does one rule out every possible hidden variable theory? Here's how: we will consider an extravagant framework within which every possible hidden variable theory. And then we will show that there is a particular quantum mechanical experiment using Bell states, whose results cannot be duplicated by any theory in this framework. The framework is this: when the Bell state is created, the two particles each make up a (infinitely long!) list of all possible experiments that they might be subjected to, and decide how they will behave under each such experiment. When the two particles separate and can no longer communicate, they consult their respective lists to coordinate their actions.

2 Bell's Thought Experiment

Bell considered the following experiment: the two particles in a Bell pair move in opposite directions to two distant apparatus. A decision about which of two experiments is to be performed at each apparatus is made randomly at the last moment, so that speed of light considerations rule out information about the choice at one apparatus being transmitted to the other. How correlated can the outcomes on the two experiments be? It can be shown that any theory in the classical hidden variable framework above gives a correlation of at most 0.75 whereas the quantum experiments described below give a correlation of about 0.8. Therefore the predictions of quantum mechanics are not consistent with any local hidden variable theory. We now describe the experiment in more detail.

The two experimenters A and B (for Alice and Bob) each receive a random bit r_A and r_B respectively. Each also receives one half of a Bell state, and makes a suitable measurement described below based on the received random bit. Call the outcomes of the measurements a and b respectively. We are interested in the achievable correlation between the two quantities $r_A \times r_B$ and $a + b(mod 2)$. We will show that for the particular quantum measurements described below $P[r_A \times r_B = a + b(mod 2)] \approx .8$.

What would a classical hidden variable theory predict for this setting? Now, when the Bell state was created, the two particles could share an arbitrary amount of information. But by the time the random bits r_A and r_B are generated, the two particles are too far apart to exchange information. Thus in any experiment, the outcome can only be a function of the previously shared information and one of the random bits. It can be shown that in this setting the best correlation is achieved by always letting the outcomes of the two experiments be $a = 0$ and $b = 0$ (see homework exercise). This gives $P[r_A \times r_B = a + b(mod 2)] \leq .75$. This experiment therefore distinguishes between the predictions of quantum physics and those of any arbitrary local hidden variable theory. It has now been performed in several different ways, and the results are consistent with quantum physics and inconsistent with any classical hidden variable theory.

Here is the protocol:

- if $X_A = 0$, then Alice measures in the $-\pi/16$ basis.
- if $X_A = 1$, then Alice measures in the $3\pi/16$ basis.
- if $X_B = 0$, then Bob measures in the $\pi/16$ basis.

- if $X_B = 1$, then Bob measures in the $-3\pi/16$ basis.

Now an easy calculation shows that in each of the four cases $X_A = X_B = 0$, etc, the success probability is $cos^2\pi/8$. This is because in the three cases where $x_A \cdot x_B = 0$, Alice and Bob measure in bases that differ by $/pi/8$. In the last case they measure in bases that differ by $3\pi/8$, but in this case they must output different bits.

We still have to prove that Bell state $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = \frac{1}{\sqrt{2}}(|v_A v_A\rangle + |v_A^\perp v_A^\perp\rangle)$ Let $|v_A\rangle = \alpha|0\rangle + \beta|1\rangle$ and $|v_A^\perp\rangle = -\beta|0\rangle + \alpha|1\rangle$. Then $\frac{1}{\sqrt{2}}(|v_A v_A\rangle + |v_A^\perp v_A^\perp\rangle) = \frac{1}{\sqrt{2}}(\alpha^2|00\rangle + \alpha\beta|01\rangle + \alpha\beta|10\rangle + \beta^2|11\rangle) + \frac{1}{\sqrt{2}}(\beta^2|00\rangle - \alpha\beta|01\rangle - \alpha\beta|10\rangle + \alpha^2|11\rangle) = \frac{1}{\sqrt{2}}(\alpha^2 + \beta^2)(|00\rangle + |11\rangle) = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$