CS 191 Lecture Notes 9 September 2008

Outline

1. Physics of quantum 2-level systems
   - Stern–Gerlach Experiment
   - Bloch Sphere representation.
   - Pauli Spin Matrices
2. Physical realizations of qubits
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   - Requirements for qubits
   - Example and model qubits
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     - Quantum harmonic oscillator
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Notes

1. The quantum two-level system is, in many ways, the simplest quantum system that displays interesting behavior (this statement is very subjective!). Before diving into the physics of two-level systems, a little about the history of the canonical example: electron spin.

In 1921, Stern proposed an experiment to distinguish between Larmor's classical theory of the atom and Somerfeld's quantum theory. Each theory predicted that the atom should have a magnetic moment (ie, it should act like a small bar magnet). However, Larmor predicted that this magnetic moment could be oriented any direction in space, while Sommerfeld (with help from Bohr) predicted that the orientation could only be in one of two directions, in this case, aligned or antialigned with a magnetic field. Stern's idea was to use the fact that magnetic moments experience a linear force when placed in a magnetic field gradient. To see this, note that the energy of a magnetic dipole in a magnetic field is given by:

$$\mathcal{U} = -\vec{\mu} \cdot \vec{B}$$

Here, $\vec{\mu}$ is the vector describing the magnetic moment. The direction of the moment is like the orientation of a bar magnet. This expression for the potential energy can also be used to derive a force that acts on the dipole (another name for something that possesses a magnetic moment):

$$F = -\nabla \mathcal{U} = \vec{\mu} \cdot \nabla \vec{B}$$

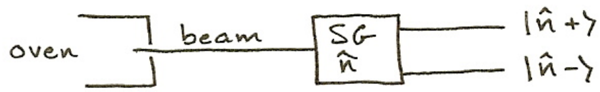
Let's suppose that the magnetic field looks like $\vec{B} = B_0 z \hat{z}$ (this field doesn't actually satisfy Maxwell's Equations, but it makes the analysis easier). This gives a force

$$F = B_0 |\mu| \cos(\theta)\, \hat{z}$$

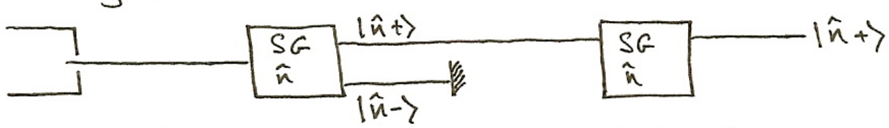
The $\cos(\theta)$ term comes from the dot product. So if the dipole is initially aligned with the field, it will experience an upward force. If it is anti-aligned, it will experience a downward force. Then a beam of these dipoles would be split by this gradient field. If they are Bohr-Sommerfeld dipoles, one expects the beam to be split by the field into two beams corresponding to the two quantized states. Larmor's theory, however, just predicts a spreading-out, because $\cos(\theta)$ can take any value from $[-1, 1]$.

In 1922, Gerlach performed this experiment using silver atoms, and he saw his beam split into two distinct beams, thus demonstrating the spatial quantization of the magnetic moment. It turns out that the Bohr-Sommerfeld theory was incorrect, and it remained untill two graduate students, Uhlenbeck and Goudsmit (1925, 1926) postulated that it was the electron that carried its own magnetic moment, independent of its orbital magnetic moment due to its being a part of some atom. So the Stern-Gerlach device was useful for distinguishing between two classes of theories: quantum vs. classical. It decidedly showed quantum behaviore, even though the precise theory it was meant to prove was ultimately shown to be wrong.

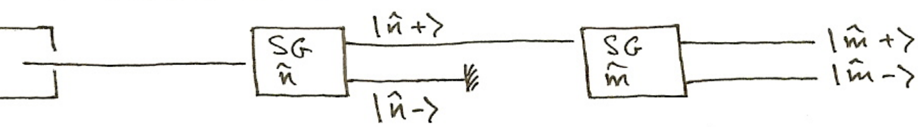
In any case, the Stern-Gerlach experiment provides a model, or toy, system that we can use to learn about quantum two level systems. Let's make our language a bit more precise by labeling some things:



In this diagram we have a cartoon picture of the Stern-Gerlach experiment. An oven produces a beam of particles which enters a region with an inhomogeneous magnetic field which has a gradient that points in the $\hat{n}$ direction. The two beams that emerge we label $|\hat{n}+\rangle$ and $|\hat{n}-\rangle$. These symbols are what we use to label a quantum state, and what is written inside gives some information about this particular state. Here, $|\hat{n}+\rangle$, is the state that is deflected upwards in a magnetic field with gradient in the $\hat{n}$ direction, while $|\hat{n}-\rangle$ is deflected downwards. Now, this gets more interesting if we consider multiple SG devices that are cascaded. Lets add another SG($\hat{n}$) after the first, but discarding the $|\hat{n}-\rangle$ state:



Notice that if we measure the output of the first SG($\hat{n}$) with a second SG($\hat{n}$) then we only get 1 beam out, the $|\hat{n}+\rangle$ state again. This shouldn't be surprising, because all we have established is that the dipole is aligned with $\hat{n}$, and remains so. But what if we rotate the second device?



Now we get two beams! The probability that $|\hat{n}+\rangle \Rightarrow |\hat{m}+\rangle$ is found experimentally to be

$$P(|\hat{n}+\rangle \Rightarrow |\hat{m}+\rangle) = \tfrac{1}{2}(1+\hat{n}\cdot\hat{m})$$

and also

$$P(|\hat{n}+\rangle \Rightarrow |\hat{m}-\rangle) = \tfrac{1}{2}(1-\hat{n}\cdot\hat{m})$$

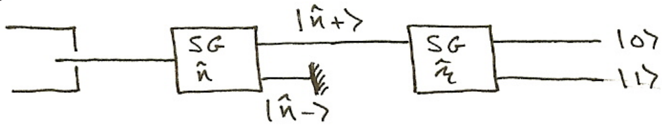
These probabilities can also be considered as the relative intensities of the two outgoing beams, $|\hat{m}+\rangle$ and $|\hat{m}-\rangle$, given an incoming beam, $|\hat{n}+\rangle$.

Our task now is to seek a quantum mechanical description of this experiment. One way to do this is to search for the simplest description we can come up with and only add complexity if we need to. Because the result of any measurement we can do is either "aligned" or "antialigned", the simplest model is the two-state system. We now consider the states $|\hat{n}\pm\rangle$ to be quantum states which are orthogonal and therefor form a basis. The states $|\hat{m}\pm\rangle$ also form a (different) basis. Let's pick $|\hat{z}\pm\rangle$ as the special basis, and express all other states in terms of linear combinations of $|\hat{z}+\rangle$ and $|\hat{z}-\rangle$. To indicate that this basis is special, let's relabel $|\hat{z}+\rangle \longrightarrow |0\rangle$ and $|\hat{z}-\rangle \longrightarrow |1\rangle$.

Now, because the set $\{|0\rangle, |1\rangle\}$ form a basis, we can express any other state as

$$|\hat{n}+\rangle = \alpha|0\rangle + \beta|1\rangle$$

Where $\{\alpha, \beta\}$ are complex numbers. I haven't talked about $|\hat{n}-\rangle$, but we'll get to that soon. Now that we know (roughly) how to express our states, we need to figure out what α and β are. We can start by considering a double Stern Gerlach experiment:



So, given the state $|\hat{n}+\rangle$, the probability of measuring $|0\rangle = |\hat{q}+\rangle$ is found to be $P(|\hat{n}+\rangle \to |0\rangle) = \frac{1}{2}(1 + \hat{n}\cdot\hat{q})$. But by the postulates of quantum mechanics, this is also given by:

$$P(|\hat{n}+\rangle \to |0\rangle) = |\langle 0|\hat{n}+\rangle|^2 = |\alpha\langle 0|0\rangle + \beta\langle 0|1\rangle|^2 = |\alpha|^2 = \frac{1}{2}(1 + \hat{n}\cdot\hat{q})$$

Here we've used the fact that $\{|0\rangle, |1\rangle\}$ form an orthonormal basis to say that the inner product $\langle 0|0\rangle = 1$ and that $\langle 0|1\rangle = 0$. Now, to make this look a bit nicer, we are going to rewrite $\hat{n}\cdot\hat{q} = \cos\theta$, where θ is the angle between the two unit vectors $\hat{n}$ and $\hat{q}$. This angle is also equal to the spherical coordinate, θ, that corresponds to $\hat{n}$:

$$\hat{n} = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$$

Applying a trig identity, $\frac{1}{2}(1 + \cos(\theta)) = \cos^2(\theta/2)$, we can say that

$$|\alpha|^2 = \cos^2(\theta/2)$$

If we note that θ runs $0 \to \pi$, then $\cos(\theta/2)$ is always positive, so

$$|\alpha| = \cos(\theta/2)$$

A similar analysis shows that $|\beta| = \sin(\theta/2)$. Using this simple arguement has determined the magnitudes of α, β. But what about their phases? Because they are complex, $\{\alpha, \beta\}$ can be written as:

$$\alpha = |\alpha| e^{i\psi} \qquad \text{and} \qquad \beta = |\beta| e^{i\chi}$$

so,

$$\begin{aligned} |\hat{n}\rangle &= |\alpha| e^{i\psi}|0\rangle + |\beta| e^{i\chi}|1\rangle \\ &= \cos(\theta/2) e^{i\psi}|0\rangle + \sin(\theta/2) e^{i\chi}|1\rangle \end{aligned}$$

But any quantum state is physically ~~the~~ unchanged by multiplication by a phase. multiplying the above expression by $\exp(-i\psi)$ gives

$$|\hat{n}+\rangle = \cos(\theta/2)|0\rangle + \sin(\theta/2) e^{i\varphi}|1\rangle \qquad \text{w/} \quad \varphi = \chi - \psi$$

So what is φ for a given $\hat{n}$? Lets look at $|\hat{x}+\rangle$ and $|\hat{y}+\rangle$:

$$|\hat{x}+\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}} e^{i\varphi_x}|1\rangle \qquad |\hat{y}+\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}} e^{i\varphi_y}|1\rangle$$

If a double stern Gerlach device is setup with $\hat{x}$ first, then $\hat{y}$, the probability of seeing $|\hat{y}+\rangle$ is $P(x \to y) = \frac{1}{2}(1 + \hat{x}\cdot\hat{y}) = \frac{1}{2}$. But it must also be

$$\begin{aligned} P(x \to y) &= |\langle \hat{x}+|\hat{y}+\rangle|^2 = \left|\left(\tfrac{1}{\sqrt{2}}\right)^2\langle 0|0\rangle + \left(\tfrac{1}{\sqrt{2}}\right)^2 e^{i(\varphi_y - \varphi_x)}\langle 1|1\rangle\right|^2 \\ &= \frac{1}{4}\left|1 + e^{i(\varphi_y - \varphi_x)}\right|^2 = \frac{1}{4}\left(1 + e^{i(\varphi_y - \varphi_x)}\right)\left(1 + e^{-i(\varphi_y - \varphi_x)}\right) \\ &= \frac{1}{4}\left(2 + 2\cos(\varphi_y - \varphi_x)\right) = \frac{1}{2} + \frac{1}{2}\cos(\varphi_y - \varphi_x) \end{aligned}$$

But this implies that $(\varphi_y - \varphi_x)$ is the angle between $\hat{x}$ and $\hat{y}$ (in this case, $\pi/2$). If we identify $\varphi_x = 0$, $\varphi_y = \pi/2$, then we see that φ can be identified with the second spherical coordinate, φ:

$$\hat{n} = (\sin\theta\cos\varphi,\ \sin\theta\sin\varphi,\ \cos\theta)$$

So, for any vector $\hat{n}$, $|\hat{n}+\rangle = \cos(\theta/2)|0\rangle + \sin(\theta/2)e^{i\varphi}|1\rangle$, where (θ,φ) are the polar coordinates of the vector $\hat{n}$. But what about $|\hat{n}-\rangle$? The states $|\hat{n}+\rangle$ and $|\hat{n}-\rangle$ must be orthogonal, so if $|\hat{n}-\rangle = \gamma|0\rangle + \delta|1\rangle$,

$$\begin{aligned}\langle\hat{n}+|\hat{n}-\rangle &= \gamma\cos(\theta/2)\langle 0|0\rangle + \delta\sin(\theta/2)e^{-i\varphi}\langle 1|1\rangle \\ &= \gamma\cos(\theta/2) + \delta\sin(\theta/2)e^{-i\varphi} \\ &= 0\end{aligned}$$

A solution to this is $\gamma = \sin(\theta/2)$, $\delta = -\cos(\theta/2)e^{i\varphi}$:

$$|\hat{n}-\rangle = \sin(\theta/2)|0\rangle + (-\cos(\theta/2))e^{i\varphi}|1\rangle$$

But compare this to $|(-\hat{n})+\rangle$. If $\hat{n}\to -\hat{n}$, then $\theta \to \pi-\theta$ and $\varphi\to\pi+\varphi$:

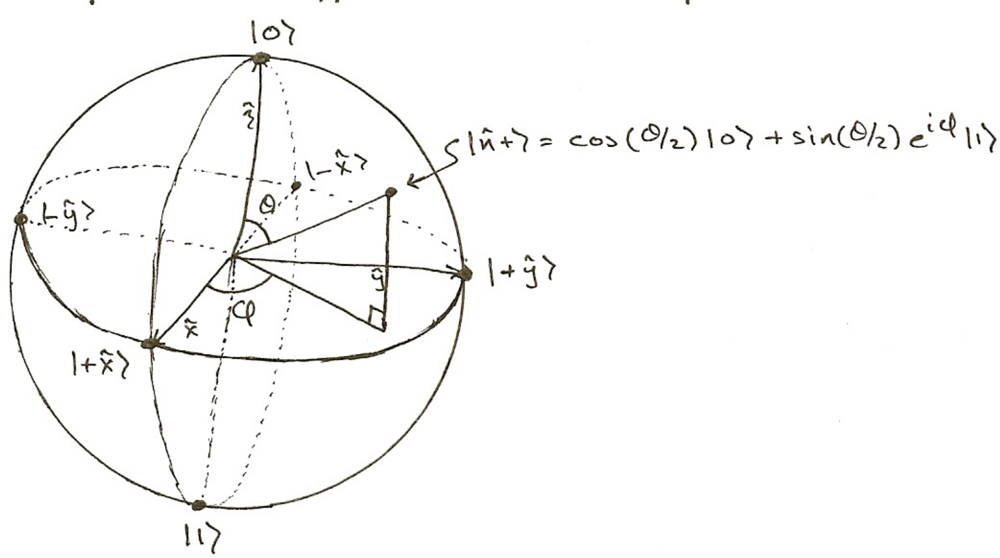
$$\begin{aligned}|(-\hat{n})+\rangle &= \cos\left(\tfrac{\pi-\theta}{2}\right)|0\rangle + \sin\left(\tfrac{\pi-\theta}{2}\right)e^{i\pi+i\varphi}|1\rangle \\ &= \sin(\theta/2)|0\rangle - \cos(\theta/2)|1\rangle \\ &= |\hat{n}-\rangle\end{aligned}$$

So $|\hat{n}+\rangle$ is orthogonal to $|(-\hat{n})+\rangle$.

This representation we have been using (θ,φ as parameters for 1-qubit states) is known as the Bloch-Sphere representation. Every point on the Bloch Sphere (a unit sphere in $\mathbb{R}^3$) corresponds to a 1-qubit state, with the orthogonal state being represented by the point on the opposite side of the sphere.

The Bloch-Sphere

antipodal points are orthogonal states



We'll see a lot more of the Bloch Sphere representation over the course of the semester, so you'll have time to get acustomed to it.

Now that we've come up with a good way to represent the states in the Stern-Gerlach experiment, we need to come up with some way to mathematically describe the Stern-Gerlach devices (what we've been calling $SG(\hat{n})$). Before we can do this, however, we need to understand a little bit about how we measure quantum states...

~~When one measures something (classically or quantumly), the goal of that measurement is to determine the a property assign a value to a particular as~~

Measurement is the assignment of a particular value to some attribute of the system under study. For instance, if you were to measure me (I'm the system), you would assign the value "red" to the attribute "hair color". One of the postulates of quantum mechanics is that for every possible measurement you can do, there exists an hermitian matrix (more properly, an hermitian <u>operator</u> which is represented frequently by a matrix). [If a matrix, A, is hermitian, then $A = A^{*T} \equiv A^{\dagger}$. Those symbols mean: '$*$' complex-conjugate, '$T$' transpose, '$\dagger$' conjugate-transpose.] The outcomes that are possible for the measurement are the eigenvalues of the operator. It might help to see an example:

Let's say we have some state: $|\hat{n}+\rangle = \cos(\theta/2)|0\rangle + \sin(\theta/2)e^{i\varphi}|1\rangle$, and we want to measure its magnetic moment in the $\hat{z}$ direction. We know from the discussion above that we should get one of two values, "aligned" or "antialligned". Because we are building a mathematical theory, lets call ~~calling~~ "aligned" $\equiv 1$ and "antialigned" $= -1$. So now we have our eigenvalues. The eigenstates that correspond to these eigenvalues are the states $|\hat{z}+\rangle = |0\rangle$ and $|\hat{z}-\rangle = |1\rangle$. In order to represent our operators as matrices, we need to represent ourstates as vectors:

$$|\hat{z}+\rangle = |0\rangle \longrightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad\qquad |\hat{z}-\rangle = |1\rangle \longrightarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

In this notation:

$$|\hat{n}+\rangle \longrightarrow \begin{pmatrix} \cos(\theta/2) \\ \sin(\theta/2)e^{i\varphi} \end{pmatrix} \qquad\qquad |\hat{n}-\rangle = |(-\hat{n})+\rangle \longrightarrow \begin{pmatrix} \sin(\theta/2) \\ -\cos(\theta/2)e^{i\varphi} \end{pmatrix}$$

So, our task now is to find a matrix, $S(\hat{z})$, that has eigenvalues $\{+1,-1\}$ and eigenvectors $\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\}$. This case is easy by inspection:

$$S(\hat{z}) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Now we need to find $S(\hat{n})$, the matrix corresponding to a measurement along $\hat{n}$. The eigenvalues are still $\{+1,-1\}$, but the eigenvectors have changed. We have already written down the eigenvectors, $\{|\hat{n}\rangle, |-\hat{n}\rangle\}$, so we can use the eigenvalue decomposition of a matrix:

$$S = P \Lambda P^{-1} \quad \text{w/} \quad \begin{aligned} \Lambda &= \text{diagonal matrix of eigenvalues} \\ P &= \text{matrix of eigenvectors} \end{aligned}$$

Let's use this to explicitly construct $S(\hat{x})$. The eigenvectors are

$$|\hat{x}+\rangle \longrightarrow \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \qquad\qquad |\hat{x}-\rangle \longrightarrow \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}$$

So:

$$P_x = \left(\begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix} \right) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \qquad\qquad \Lambda_x = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Then

$$S(\hat{x}) = P_x \Lambda P_x^{-1} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \frac{1}{\sqrt{2}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

This case was alittle special, because $P_x = P_x^{-1}$. Doing the same for $S(\hat{y})$

$$P_y = \left(\begin{pmatrix} 1/\sqrt{2} \\ i/\sqrt{2} \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} \\ -i/\sqrt{2} \end{pmatrix} \right) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \Rightarrow S(\hat{y}) = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

These three matrices that we've just derived are very special. So special, in fact, that we are going to give them special names:

$$S(\hat{x}) \longrightarrow \sigma_x \equiv X \qquad\qquad \sigma_x \text{ if you're a Physicist}$$
$$S(\hat{y}) \longrightarrow \sigma_y \equiv Y \qquad\qquad X \text{ if you're a Computer Scientist}$$
$$S(\hat{z}) \longrightarrow \sigma_z \equiv Z$$

They are called the Pauli Matrices, and you'll see them a lot this semester. A nice feature of them is that any matrix $S(\hat{n})$ can be written as

$$S(\hat{n}) = \hat{n}\cdot\vec{\sigma} \equiv n_x\sigma_x + n_y\sigma_y + n_z\sigma_z$$

This is a nice exercise, and I recommend you try to show that it's true. So, we have these matrices, how do we use them? Let's say we have the state $|\hat{n}\rangle$ and we want to measure it using $S(\hat{x})$, and determine the probabilities of seeing "aligned" or "antialigned". To do this, we express

$$|\hat{n}\rangle = \mu|\hat{x}+\rangle + \nu|\hat{x}-\rangle$$

Multiply on the left by $\langle\hat{x}+|$,

$$\langle\hat{x}+|\hat{n}\rangle = \mu\langle\hat{x}+|\hat{x}+\rangle + \nu\langle\hat{x}+|\hat{x}-\rangle = \mu\cdot 1 + \nu\cdot 0 = \mu$$

So,
$$\mu = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}^T \cdot \begin{pmatrix} \cos(\theta/2) \\ \sin(\theta/2)e^{i\varphi} \end{pmatrix} = \langle\hat{x}+|\hat{n}\rangle$$
$$= \frac{1}{\sqrt{2}}\left(\cos(\theta/2) + \sin(\theta/2)e^{i\varphi}\right)$$

Likewise
$$\nu = \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}^T \cdot \begin{pmatrix} \cos(\theta/2) \\ \sin(\theta/2)e^{i\varphi} \end{pmatrix} = \langle\hat{x}-|\hat{n}\rangle$$
$$= \frac{1}{\sqrt{2}}\left(\cos(\theta/2) - \sin(\theta/2)e^{i\varphi}\right)$$

Then the state $|\hat{n}\rangle$ is:

$$|\hat{n}\rangle = \frac{1}{\sqrt{2}}\left(\cos(\theta/2) + \sin(\theta/2)e^{i\varphi}\right)|\hat{x}+\rangle + \frac{1}{\sqrt{2}}\left(\cos(\theta/2) - \sin(\theta/2)e^{i\varphi}\right)|\hat{x}-\rangle$$

And the probability of measuring "aligned" is $|\mu|^2 = \frac{1}{2}\left|\cos(\theta/2) + \sin(\theta/2)e^{i\varphi}\right|^2$ or,

$$|\mu|^2 = \frac{1}{2}\left(1 + \sin(\theta/2)\cos(\theta/2)(2\cos\varphi)\right)$$
$$= \frac{1}{2}\left(1 + \sin(\theta)\cos(\varphi)\right)$$

And
$$|\nu|^2 = \frac{1}{2}\left(1 - \sin(\theta)\cos(\varphi)\right)$$