Quantum Simulation
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Quantum Simulation

- Studies the efficient classical simulation of quantum circuits
- Understanding what is (and is not) efficiently simulable has implications to the usefulness of quantum circuits
- Known to be closely related to entanglement
- In general, simulation is exponential but some classes of circuits may be done efficiently
Three Results in Quantum Simulation

- Any poly sized (and generally poly depth) circuit on n-qubits can be efficiently simulated provided the maximum number of gates touching or spanning any wire is bounded by $O(\log n)$. (Jozsa, 2006)
- The evolution of a quantum state can be simulated with resources linear in the size of the state and exponential in the entanglement. If the maximum Schmidt-rank of the system is poly in the size of the state, then the simulation is efficient. (Vidal, 2003)
- Multi-scale Entanglement Renormalization Ansatz (MERA) is a way of describing a quantum circuit as a lattice with special properties that make it classically simulable (Vidal, 2006)
Quantum Circuit Reduction

- Reduce quantum circuits to having only 1- and 2-qubit gates
- Formulate as parallel wires, and reduce it by merging all 1-qubit gates into neighboring gates, and merging all consecutive 2-qubit gates that can be performed sequentially without an interposing gate.

A circuit $C$

The reduced form $C_{\text{red}}$
Contracting Linear Networks

- For each wire, let $D_i$ be the number of 2-qubit gates that touch or cross it. Then, $D = \max_i D_i$.

$$A^{i_3j_4k_3l_2} = b^{i_1} b^{j_1} b^{k_1} b^{l_1} U^{i_2j_2} V^{j_3k_2} W^{i_3j_4} X^{l_2k_3}$$
Contracting Linear Networks (efficient computation)

- We measure with the projector matrix $\Pi_n^m = |k\rangle\langle k|$.

- The output is prob($k$) can be easily computed from the full transformation.

- This sum takes time $O(2^D)$, so doing so for all wires takes time $O(n \cdot 2^D)$ which is polynomial in $n$ if $D = O(\log n)$. 
Schmidt Decomposition

- As a consequence from linear algebra (related to SVD) we can decompose any quantum state $|\Psi\rangle \in \mathcal{H}_2^\otimes n$ into two subspaces $\mathcal{H}_2^\otimes n = \mathcal{H}_A \otimes \mathcal{H}_B$ where:

$$|\psi\rangle_{AB} = \sum_{i=1}^{n} \tilde{\chi}_i |u_i\rangle_A \otimes |v_i\rangle_B$$

- The number of non-zero Schmidt coefficients ($\tilde{\chi}_i$) is the Schmidt rank of the decomposition.
- The state is separable iff it has a Schmidt rank of one.
- If more than one coefficient is non-zero, the state is entangled. If they are all non-zero, it is maximally entangled.
- Thus the maximum Schmidt rank makes a good measure of the state's entanglement.
Slightly entangled states

"We say a pure-state quantum evolution is slightly entangled if, at all times \( t \), the state \( |\Psi(t)\rangle \) of the system is slightly entangled – that is, if \( \chi(t) \) is small. A sequence of evolutions with an increasingly large number of \( n \) qubits is slightly entangled if \( \chi_n(t) \) is upper bounded by \( \text{poly}(n) \)." (Vidal, 2003)

- By restricting ourselves to circuits with limited entanglement, we can prove them to be efficiently simulable.
- Conversely, we can place a lower bound on the entanglement in useful quantum circuits.
Local state decomposition

- In general, representing $|\Psi\rangle \in \mathcal{H}_2^\otimes n$ requires exponential state. If it is only slightly entangled, we can inductively use Schmidt decomposition to get:

$$|\Psi\rangle \leftrightarrow \Gamma^{[1]} \lambda^{[1]} \Gamma^{[2]} \lambda^{[2]} \ldots \Gamma^{[l]} \ldots \lambda^{[n-1]} \Gamma^{[n]} \quad (2\chi^2 + \chi)n$$

- Which is specified in terms of parameters.
- This representation requires we pick an ordering of qubits, but once we've done so we can keep updates local. That is to apply a gate to qubits $l$ and $l+1$ we need only update $\Gamma^{[l]}$, $\lambda^{[l]}$, and $\Gamma^{[l+1]}$. 
Mind the gap

- Problem: Our decomposed representation allows for efficient local update, but gates can span multiple wires.
- Solution: decompose a gate spanning $r$ wires into $O(r)$ gates spanning two adjacent wires.
Efficient updates

- Single qubit gates only transform $\Gamma^{[l]}$ and can be computed in $O(\chi^2)$ time.
- Proof sketch: consider that a unitary operation $U$ doesn't affect the Schmidt decomposition above or below $l$.
- By a similar (but more involved) argument, it is possible to show that applying a gate to $l$ and $l+1$ can be done in $O(\chi^3)$ time and $O(\chi^2)$ space.
- Thus, provided $\chi$ is constrained to grow slowly, the circuit can be efficiently simulated.
- Note: global entanglement appears key to computational speed up.
Multi-scale Entanglement Renormalization Ansatz (MERA)

- A MERA efficiently encodes a lattice of quantum states of spatial dimension, D.
- The MERA for $|\Psi\rangle$ is a tensor network, $M$, that corresponds to quantum circuit, $C$.
- The dimension of the q-states in $|\Psi\rangle$ is $X$. 
C[s] includes the gates and wires that influence p[s]

Ctheta[s], the number of wires in a time slice of C[s], is bounded by \(3^{(D-1)}*4\)
MERAs: Computing $p[s]$ and $p[s1s2]$

- The density matrix for each slice of $C\theta[s]$ and $C\theta[s1s2]$ can be computed from the density matrix for the previous slice in time polynomial to $X$.
- Since there are $\log N$ slices, $p[s]$ and $p[s1s2]$ (generally: $p[s1...sn]$) can be computed in time $O(X*\log N)$. 
Conclusions

- Quantum simulation is (necessarily) hard
- Understanding what can be efficiently simulated places lower bounds on worthwhile quantum circuits
- Entanglement appears to be the limiting factor in simulation
- Better models constraining entanglement enable efficient algorithms; the mathematical models can be thought of as efficient data-structures.
Questions?