1. Let $f : \{0,1\}^n \to \{0,1\}$. Let U_f be the unitary transformation (quantum circuit) that on input $|x\rangle|b\rangle$ (where $x \in \{0,1\}^n$ and $b \in \{0,1\}$), outputs $|x\rangle|b \oplus f(x)\rangle$. Show that if the answer bit b is initialized to $|-\rangle$ then then on input $1/\sqrt{2}|x\rangle|-\rangle$, the output is $\frac{(-1)^{f(x)}}{\sqrt{2}}|x\rangle|-\rangle$. i.e. the answer bit remains unchanged, and effectively the transformation that is achieved is: $|x\rangle \to (-1)^{f(x)}|x\rangle$.

2. Let $f : \{0,1\}^n \to \{0,1\}^n$ be a bijection on the n-bit strings. We showed in class that if there is an efficient classical circuit (say of size m) for computing $f(x)$ on input x, then there is an efficient reversible circuit (of size $O(m)$) that outputs $x \cdot f(x) \cdot 0^k$ on input $x \cdot 0^{n+k}$, where $k = O(m)$.

   Now suppose that you are also given an efficient classical circuit for computing f^{-1} (of size m'). Show that there is an efficient reversible circuit that outputs $f(x) \cdot 0^l$ on input $x \cdot 0^l$. What is the size of your circuit and how big is l?

3. Consider the state $|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. What is the state that results from applying the 2-qubit Hadamard transform to $|\psi\rangle$?

   Repeat for the state $|\phi\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$.

4. Consider a superposition on n bit strings $|\psi\rangle = \frac{1}{\sqrt{M}}\sum_{x:u\cdot x=0}|x\rangle$ where u is a fixed non-zero n bit string and $u \cdot x = u_1x_1 + u_2x_2 \ldots u_nx_n (mod\ 2)$. What is M?

   Apply the n-qubit Hadamard transform to $|\psi\rangle$ and let the result be $|\phi\rangle = \sum_x \alpha_x|x\rangle$. What is $\alpha_{00\ldots0}$? What is $|\phi\rangle$

5. What is the quantum fourier transform modulo M of the uniform superposition $\frac{1}{\sqrt{M}}\sum_{j=0}^{M-1}|j\rangle$?

   What is the QFT modulo M of $|j\rangle$?

6. Let $|\beta\rangle = \frac{1}{2^{n/2}}\sum_{x\in\{0,1\}^n}\beta_x|x\rangle$ be the Hadamard transform of the superposition $|\alpha\rangle = \frac{1}{2^{n/2}}\sum_{x\in\{0,1\}^n}\alpha_x|x\rangle$

   Consider the 'shift' by u of the superposition $|\alpha\rangle$: $|\alpha'\rangle = \frac{1}{2^{n/2}}\sum_{x\in\{0,1\}^n}\alpha_x|x+u\rangle$ (where $x+u$ means bitwise sum modulo 2) and its Hadamard transform $|\beta'\rangle = \frac{1}{2^{n/2}}\sum_{x\in\{0,1\}^n}\beta'_x|x\rangle$. Describe $|\beta'\rangle$ as a function of $|\beta\rangle$ and u.