Tensor Products

1 Tensor Products

Consider a two qubit system. Each qubit lives in a 2-dimensional complex vector space (also called a Hilbert space) $\mathscr{C}^2$. But as we have seen, the state of a two qubit system lives in a 4-dimensional Hilbert space $\mathscr{C}^4$. How do we combine two copies of $\mathscr{C}^2$ to get $\mathscr{C}^4$? This is formally done by taking tensor products. We will not get into all the mathematical details of this operation, but instead describe just enough to make sense of it.

We denote the tensor product of the Hilbert spaces for the two qubits by $\mathscr{C}^2 \otimes \mathscr{C}^2$ (read C2 tensor C2). It turns out that $\mathscr{C}^2 \otimes \mathscr{C}^2$ is isomorphic to $\mathscr{C}^4$.

If V, W are vector spaces with orthonormal bases $\{v_1 \dots v_n\}$, $\{w_1 \dots w_m\}$, the *tensor product* $V \otimes W$ of V and W is a nm-dimensional vector space which has an orthonormal basis $\{v_i \otimes w_j \ : \ 1 \leq i \leq n \ , \ 1 \leq j \leq m\}$. Each element of the form $v \otimes w$ is called an *elementary tensor*.

This is a bit abstract, so let us consider an example: we get an orthonormal basis for $\mathscr{C}^2 \otimes \mathscr{C}^2$ by taking the basis vectors $\{\left|0\right\rangle, \left|1\right\rangle\}$ and considering the set of elementary tensors:

$$\{|0\rangle \otimes |0\rangle, |0\rangle \otimes |1\rangle, |1\rangle \otimes |0\rangle, |1\rangle \otimes |1\rangle\}$$

We will often write $|0\rangle \otimes |0\rangle$ as $|0\rangle|0\rangle$ or $|00\rangle$.

In general, we represent an n-particle system by n copies of $\mathscr{C}^2$ tensored together. We will often write $(\mathscr{C}^2)^{\otimes n} = \mathscr{C}^{2^n}$. So the state of an n-qubit system can be written as

$$|\psi\rangle = \sum_{x \in \{0,1\}^n} \alpha_x |x\rangle.$$

This means that the state of an n-particle system is represented by a 2^n dimensional space! The idea behind quantum computation is to harness the ability of nature to manipulate the exponential number of α_xs.

1.1 The Significance of Tensor Products

Classically, if we put together a subsystem that stores k bits of information with one that stores l bits of information, the total capacity of the composite system is $k+l$ bits.

From this viewpoint, the situation with quantum systems is extremely paradoxical. We need k complex numbers to describe the state of a k-level quantum system. Now consider a system that consists of a k-level subsystem and an l-level subsystem. To describe the composite system we need kl complex numbers. One might wonder where nature finds the extra storage space when we put these two subsystems together.

An extreme case of this phenomenon occurs when we consider an n qubit quantum system. The Hilbert space associated with this system is the n-fold tensor product of $\mathscr{C}^2 \equiv \mathscr{C}^{2^n}$. Thus nature must "remember" of 2^n complex numbers to keep track of the state of an n qubit system. For modest values of n of a few hundred, 2^n is larger than estimates on the number of elementary particles in the Universe.

This is the fundamental property of quantum systems that is used in quantum information processing.

Finally, note that when we actually a measure an n-qubit quantum state, we see only an n-bit string - so we can recover from the system only n, rather than 2^n, bits of information.

1.2 Tensor product of operators

Suppose $|v\rangle$ and $|w\rangle$ are unentangled states on $\mathscr{C}^m$ and $\mathscr{C}^n$, respectively. The state of the combined system is $|v\rangle \otimes |w\rangle$ on $\mathscr{C}^{mn}$. If the unitary operator A is applied to the first subsystem, and B to the second subsystem, the combined state becomes $A|v\rangle \otimes B|w\rangle$.

In general, the two subsystems will be entangled with each other, so the combined state is not a tensor-product state. We can still apply A to the first subsystem and B to the second subsystem. This gives the operator $A \otimes B$ on the combined system, defined on entangled states by linearly extending its action on unentangled states.

(For example, $(A\otimes B)(|0\rangle \otimes |0\rangle) = A|0\rangle \otimes B|0\rangle$. $(A\otimes B)(|1\rangle \otimes |1\rangle) = A|1\rangle \otimes B|1\rangle$. Therefore, we define $(A\otimes B)(\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle)$ to be $\frac{1}{\sqrt{2}}(A\otimes B)|00\rangle + \frac{1}{\sqrt{2}}(A\otimes B)|11\rangle = \frac{1}{\sqrt{2}}\left(A|0\rangle \otimes B|0\rangle + A|1\rangle \otimes B|1\rangle\right)$.)

Let $|e_1\rangle, \ldots, |e_m\rangle$ be a basis for the first subsystem, and write $A = \sum_{i,j=1}^m a_{ij}|e_i\rangle\langle e_j|$ (the i,jth element of A is a_{ij}). Let $|f_1\rangle, \ldots, |f_n\rangle$ be a basis for the second subsystem, and write $B = \sum_{k,l=1}^n b_{kl}|f_k\rangle\langle f_l|$. Then a basis for the combined system is $|e_i\rangle \otimes |f_j\rangle$, for $i = 1, \ldots, m$ and $j = 1, \ldots, n$. The operator $A \otimes B$ is

$$\begin{aligned} A\otimes B &= \left(\sum_{ij} a_{ij}|e_i\rangle\langle e_j|\right) \otimes \left(\sum_{kl} b_{kl}|f_k\rangle\langle f_l|\right) \\ &= \sum_{ijkl} a_{ij}b_{kl}|e_i\rangle\langle e_j| \otimes |f_k\rangle\langle f_l| \\ &= \sum_{ijkl} a_{ij}b_{kl}(|e_i\rangle \otimes |f_k\rangle)(\langle e_j| \otimes \langle f_l|) \ . \end{aligned}$$

Therefore the $(i,k),(j,l)$th element of $A \otimes B$ is $a_{ij}b_{kl}$. If we order the basis $|e_i\rangle \otimes |f_j\rangle$ lexicographically, then the matrix for $A \otimes B$ is

$$\begin{pmatrix} a_{11}B & a_{12}B & \cdots \\ a_{21}B & a_{22}B & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} \ ;$$

in the i,jth subblock, we multiply a_{ij} by the matrix for B.