1. Entropy as a measure of information

Suppose we wish to quantify just how much information is provided by an event \( E \) which may occur in a probabilistic experiment, or alternatively, how much information do we acquire by witnessing the event \( E \). We do this using an information function \( f(E) \) whose value is determined by the event \( E \). We make the following reasonable assumptions about this function:

(a) \( f(E) \) is a function only of the probability of the event \( E \), so we may write \( f = f(p) \), where \( p \) is a probability in the range 0 to 1.

(b) \( f \) is a smooth function of probability.

(c) When two independent events occur with individual probabilities \( p > 0 \) and \( q > 0 \), the information gained is given by the sum of the information gained from each event alone, i.e., \( f(pq) = f(p) + f(q) \), i.e, information from independent events is additive.

Show that these relationships imply that \( f(p) = k \log p \), for some constant, \( k \). Hence show that the average information gain when one of a mutually exclusive set of events with probabilities \( p_1, p_2, ..., p_n \) occurs is \( k \sum_i p_i \log p_i \). Disregarding the constant factor \( k \), this is the Shannon (classical) entropy, which is usually written as \( H(p) = -\sum_i p_i \log p_i \). How would you determine the sign of \( k \)?

2. Entanglement and robustness of three-qubit states

Consider the two 3-qubit states:

- GHZ states: \( \frac{1}{\sqrt{3}} (|000\rangle + |111\rangle) \)
- W states: \( \frac{1}{\sqrt{3}} (|100\rangle + |010\rangle + |001\rangle) \)

(a) Compare the robustness of the GHZ state and the W state to loss of information about a single qubit (corresponding to either non-recorded (unreferred) measurement or physical loss of the qubit).

(b) Find the entanglement entropy of the GHZ state and of the W state.

(c) Compare your answers for the two states with regard to robustness and degree of entanglement.