1 Stern-Gerlach Apparatus

A Stern-Gerlach device is simply a magnet set up to generate a particular inhomogeneous \( \vec{B} \) field.

When a particle with spin state \( |\psi\rangle = \alpha |0\rangle + \beta |1\rangle \) is shot through the apparatus from the left, its spin-up portion is deflected upward, and its spin-down portion downward. The particle’s spin becomes entangled with its position! Placing detectors to intercept the outgoing paths therefore measures the particle’s spin.

Why does this work? We’ll give a semiclassical explanation – mixing the classical \( \vec{F} = m \vec{a} \) and the quantum \( H |\psi\rangle = E |\psi\rangle \) – which is quite wrong, but gives the correct intuition. [See Griffith’s § 4.4.2, pp. 162-164 for a more complete argument.] Now the potential energy due to the spin interacting with the field is

\[ E = -\vec{\mu} \cdot \vec{B}, \]

so the associated force is

\[ \vec{F}_{\text{spin}} = -\nabla E = \nabla (\vec{\mu} \cdot \vec{B}). \]

At the center \( \vec{B} = B(z)\hat{z} \), with \( \frac{dB}{dz} < 0 \), so \( \vec{F} = \nabla (\mu_z B(z)) = \mu_z \frac{dB}{dz} \hat{z} \). The magnetic moment \( \vec{\mu} \) is related to spin \( \vec{S} \) by \( \vec{\mu} = \frac{g \mu_B}{2 m} \vec{S} = -\frac{e}{m} \vec{S} \) for an electron. Hence

\[ \vec{F} = \frac{e}{m} \frac{dB}{dz} S_z \hat{z}; \]

if the electron is spin up, the force is upward, and if the electron is spin down, the force is downward.

2 Initialize a Qubit

- How can we create a beam of qubits in the state \( |\psi\rangle = |0\rangle \)? Pass a beam of spin-\( \frac{1}{2} \) particles with randomly oriented spins through a Stern-Gerlach apparatus oriented along the 3 axis. Intercept the downward-pointing beam, leaving the other beam of \( |0\rangle \) qubits.

Note that we measure the spin when we intercept an outgoing beam – after this measurement, the experiment is probabilistic and not unitary.

- How can we create a beam of qubits in the state \( |\psi\rangle = \alpha |0\rangle + \beta |1\rangle \)? First find the point on the Bloch sphere corresponding to \( |\psi\rangle \). That is, write

\[ |\psi\rangle = \alpha |0\rangle + \beta |1\rangle = \cos \frac{\theta}{2} |0\rangle + e^{i\phi} \sin \frac{\theta}{2} |1\rangle \]
(up to a phase), where
\[ \tan \frac{\theta}{2} = \frac{|\beta|}{|\alpha|}, \quad e^{i\varphi} = \frac{\beta/|\beta|}{\alpha/|\alpha|}. \]
The polar coordinates \( \theta, \varphi \) determine a unit vector \( \hat{n} = \cos \varphi \sin \theta \hat{x} + \sin \varphi \sin \theta \hat{y} + \cos \theta \hat{z} \). Now just point the Stern-Gerlach device in the corresponding direction on the Bloch sphere, and intercept one of the two outgoing beams. That is, a Stern-Gerlach device pointed in direction \( \hat{n} \) measures \( S_{\hat{n}} = \hat{n} \cdot \hat{S} \).

- How can we implement a unitary (deterministic) transformation? We need to evolve the wave function according to a Hamiltonian \( \hat{H} \). Then
\[ |\psi(t)\rangle = e^{-i\hat{H}t} |\psi(0)\rangle \]
solves the Schrödinger equation (if \( \hat{H} \) is time-independent). In the next lecture we will show how to accomplish an arbitrary single-qubit unitary gate (a rotation on the Bloch sphere) by applying a precise magnetic field for some precise amount of time: Larmor precession.