A new Qubit: Electron state of an atom!

Consider an atom whose only 2 atomic states are important:

\[ \begin{align*}
    |1\rangle, E_1 \\
    |0\rangle, E_0
\end{align*} \]

\[ H_0|1\rangle = E_1|1\rangle \]
\[ H_0|0\rangle = E_0|0\rangle \]

|0\rangle, |1\rangle refer to atomic orbitals:

\[ |0\rangle, |1\rangle \rightarrow \psi_{\text{atom}} (r, \theta, \phi) \]

Atomic wavefun can be quite complex, but lets not worry about this detail (look it up in books), we just assume it exists.

\[ \text{What does } H_0 \text{ look like in basis } |0\rangle, |1\rangle ? \]

We showed it looks like \[ H_0 = \begin{pmatrix} E_0 & 0 \\ 0 & E_1 \end{pmatrix} \] Different \( E_1 - E_0 \) plays same role as \( B_0 \) did for spin!

Now consider arbitrary electronic state: \[ |\psi\rangle = \alpha|0\rangle + \beta|1\rangle \]

Can project onto Bloch Sphere. How does it change in time?

\[ |\psi(t)\rangle = e^{-iH_0 t/\hbar} |\psi\rangle \]

\[ |\psi(t)\rangle = \alpha e^{-iE_0 t/\hbar}|0\rangle + \beta e^{-iE_1 t/\hbar}|1\rangle \]

(just like spin!)

\[ \psi(t) = (e^{-iE_0 t/\hbar})(|0\rangle + \beta e^{-iE_1 t/\hbar}|1\rangle) \]

\[ W_0 = \frac{E_1 - E_0}{\hbar} \]

State vector spins around \( 7-\phi \) is

But, although \( H_0 \) causes \[ |\psi(t)\rangle \] to spin around \( \phi \)-axis, it will never cause it to change "latitude" on Bloch Sphere. i.e., no "spin flips" or transitions. How come then?
To cause us to spin around Bloch sphere at constant latitude \((\theta = \text{const.})\), just like \(\vec{B} = \vec{B}_0 \hat{z}\).

**Question:** How do we change our atomic qubit state in a way that \(\theta\) changes, where \(|\psi\rangle = \alpha |0\rangle + \beta |1\rangle = \cos \frac{\theta}{2} |0\rangle + e^{i\phi} \sin \frac{\theta}{2} |1\rangle\)?

(i.e., change latitude on Bloch sphere)

**Answer:** Need to perturb our system with a "force field" so that Hamiltonian gets off-diagonal matrix elements!! (like \(\vec{B}_0\) for spin)

\[
H = \begin{pmatrix}
H_1 & H_{12} \\
H_{12} & H_{22}
\end{pmatrix}
\]

In order to change ratio between occupancy of \(|0\rangle\) & \(|1\rangle\), we must have \(H_{12} \neq 0\)

**BUT,** \(H_{12} = 0\) for \(H_0\), so we need a **new term** in the Hamiltonian.

How do we get it? **Turn on an \(E\)-field!!**

This applies a force that causes \(H_{12} \neq 0\) and hence "vertical" rotations on Bloch sphere in exactly the same manner as \(\vec{B} = \vec{B}_1\) did for spin...

To understand this, we must understand what happens when we subject an atom to an \(E\)-field. How do we do this?? **SOLVE SCHROED Eqn.**
Solve Schröd. Eq'n!!

1st Must Find $\hat{H}$

What is classical energy of an atom in an $E$-field? Must find the energy of this "perturbation".

Consider static $E$-field:

\[ \mathbf{E} = E\mathbf{\hat{z}} \]

Exponentially:

\[ V = \frac{1}{\epsilon_0 E} \]

Only worry about electron, since it is much lighter than nucleus.

\[ \mathbf{F} = -e\mathbf{E} = -eE\mathbf{\hat{z}} \]

\[ \Rightarrow E = \int \mathbf{E} \cdot d\mathbf{r} = \int eE' d\mathbf{z} = eE'z \]

\[ \Rightarrow U = eE'z \quad \text{Potential energy depends on } z \text{-location of electron} \]

\[ = \frac{e}{2} \left[ \text{This is the energy term that causes transition between } |10\rangle \text{ and } |11\rangle \right] \]

Now,

\[ \hat{\mathbf{H}} = \hat{\mathbf{H}}_0 + \hat{\mathbf{H}}' \]

\[ \hat{\mathbf{H}}' = eE' \mathbf{\hat{z}} \]

$\hat{\mathbf{H}}'$ is called the "dipole" Hamiltonian since if we define an electric dipole $\mathbf{p} = -e\mathbf{r}$:

\[ \hat{\mathbf{H}}' = -\mathbf{p} \cdot \mathbf{E} = -(-e\mathbf{r}) \cdot E'\mathbf{\hat{z}} = eE'z \]

How does $\hat{\mathbf{H}}'$ change a $2 \times 2$ representation of $\hat{\mathbf{H}}$? and how can this be used to control $|\psi\rangle$??
To see how qubit changes under influence of perturbation, find \( \hat{H} \) and solve TDS: 

\( \hat{H} = \hat{H}_0 + eE'z \) - what does \( \hat{H} \) look like in qubit basis \( |0\rangle, |1\rangle \)?

\[
\hat{H} = \begin{pmatrix}
H_{00} & H_{01} \\
H_{10} & H_{11}
\end{pmatrix}
\]

Calculate these matrix elements:

\[
\begin{align*}
H_{00} &= |0\rangle \langle 0| + eE'z |1\rangle \langle 1| = \langle 0|H_0|0\rangle + \langle 0|eE'z|1\rangle \\
&= E_0 + \langle 0|eE'z|0\rangle
\end{align*}
\]

Find \( \langle 0|eE'z|1\rangle \) - calculate matrix element in spherical coordinates:

\[
\begin{align*}
\langle 0| e E' z | 1 \rangle &= \int_0^1 \int_0^{\pi} \int_0^{2\pi} R_{00}^* (r) Y_{0m} (0, \theta) eE'z \cos \theta R_{10} (r) Y_{1m} (0, \phi) r^2 \sin \theta d\theta d\phi \\
&= 0 \quad \text{from} \\
&= \Delta l = \pm 1 \\
&= \pm 1 \\
&= 0 \\
\end{align*}
\]

So, \( H_{00} = E_0 \)

Similarly, \( H_{11} = E_1 \)

BUT, what about the crucial \( H_{12} (= H_{21}^*) \) term?

Calculate OFF-DIAG matrix element:
\[ H_{12} = \langle 0 | H_0 + eE'z | 1 \rangle = \langle 0 | H_0 | 1 \rangle + \langle 0 | eE'z | 1 \rangle \]
\[ \Rightarrow \langle 0 | H_0 | 1 \rangle = E_1 \langle 0 | 1 \rangle = 0 \]
\[ \Rightarrow \text{Must Find} \langle 0 | eE'z | 1 \rangle \Rightarrow \text{Calculate integral:} \]
\[ \langle 0 | eE'z | 1 \rangle = \iiint \rho_{\lambda \mu} (r) \rho_{\mu \nu} (r') e^{i E'r_0 \phi} e^{i E'z \phi} \rho_{\nu \phi} (r) \rho_{\phi \phi} (r') r_0 r_0 r \, dr \, d\phi \, dr' \, d\phi' \, dr'' \, d\phi'' \]
Here, however, state \( |0\rangle \) & \( |1\rangle \) can be chosen so that \( \beta \neq 0 \) and \( \Delta l = l_2 - l_1 = \pm 1 \) so that selection rule does not make integral \( = 0 \).
\[ \Rightarrow \text{Can solve integral (use Mathematica, or look up in book)} \]
and we find that \( \langle 0 | eE'z | 1 \rangle \neq 0 \)

Assume integral is known \( \Rightarrow \text{Define:} \langle 0 | eE'z | 1 \rangle = V_1 \)

\[ H_{12} = V_1 \]
\[ H_{21} = V_1^* \]

\[ H = \begin{pmatrix} E_0 & V_1 \\ V_1^* & E_1 \end{pmatrix} \Rightarrow \text{Off-diag. matrix elements are } \neq 0 \]
\[ \Rightarrow \text{This induces transitions between } |0\rangle \text{ & } |1\rangle \text{!}

\[ \Rightarrow \text{What does this look like on Bloch Sphere?} \]
Remember, $H_0$ causes $\left| 14\right>$ to rotate about $z$-axis at $\omega_0 = \frac{E_0 - E_0}{\hbar}$

$E_1 - E_0$ plays role of $\vec{B} = \vec{B}_0 \times \hat{x}$

$E$-field plays role of $\vec{B}_L$!

Off-diag. matrix element, $V_1$, plays role of $\vec{B}_L = B_L \hat{x}$

What happens?

$B_x \approx V_1$ causes total $\vec{B}$-field to "tilt," and $\left| 14\right>$ rotate around new total "$\vec{B}$-field".

$\vec{B}_{\text{tot}} = \vec{B}_0 + B_x \hat{x} \Rightarrow \vec{B}_T \approx \vec{B}_T^{(1)}$

So, "intitude" on Bloch Sphere DOES change, i.e., $\Theta$ changes for $\left| 14\right> = \cos \frac{\omega_0 t}{\hbar} + e^{i\phi} \sin \frac{\omega_0 t}{\hbar}$

BUT, $\Theta$ will NOT change by much unless $B_x \approx B_0$. In other words, for a significant change in $\Theta$, need $V_1 \approx E_1 - E_0$

But physically this is VERY difficult.
Since it would require electric fields on the order of volts/Å, since the size of an atom is ~ 1 Å and $E_1 - E_0 \approx 1$ eV

Need capacitor plates to be very close. \( \frac{1}{H} \frac{1}{E} \frac{1}{E_0} \frac{1}{1 Å} \)

\[ \text{UNREASONABLE!} \]

So, what else can we do to control the state of our electronic qubit? ?? $\ket{14} = \cos \frac{\theta}{2} \ket{0} + e^{i \phi} \sin \frac{\theta}{2} \ket{1}$

How can we change $\theta$, or "tilt" the $\ket{14}$ vector more on Bloch?

\[ \text{Answer: Use Resonance Technique!!!} \]

Do what we did for Spin Resonance!

Oscillate the $\vec{E}$-field at the frequency $\omega_0 = \frac{E_1 - E_0}{\mathcal{L}}$.

Can control $\Theta(t)$ very precisely.

\[ \frac{\partial}{\partial t} \vec{E} = \vec{E}' \rightarrow \vec{E}(t) = \vec{E} \cos \omega_0 t \rightarrow \text{ Oscillate at $\omega_0$} \]

\[ \text{BUT, $\omega_0$ is very large!} \quad E_1 - E_0 \approx 1$eV $\rightarrow \omega_0 \approx 10^{15}$Hz.

That's the Frequency of Light! !

So, to create $\vec{E} = \vec{E} \cos \omega_0 t \hat{x}$, we don't use a capacitor plate, we shine light on the atom! !

\[ \text{Light is an oscillating $E$-field!} \]

\[ \text{Light} \rightarrow \text{Direction} \]
So, what does atomic qubit do when we shine light on it?

Must solve Schröd. Eq.\[ i\hbar \frac{\partial}{\partial t} \psi(t) = \hat{H}\psi(t) \]

\[ \hat{H} = \begin{pmatrix} E_0 & V_1 \cos \omega t \\ V_1 \cos \omega t & E_1 \end{pmatrix} \]

$V_1 \rightarrow V, \cos \omega t$

by shining light on atom, $V, \omega \propto$ light intensity

$\Rightarrow$ This is essentially the same problem that we solved before for an oscillating $E_1$ applied to spin.

Spin Hamiltonian: \[ \hat{H} = \frac{\hbar}{2m} \begin{pmatrix} B_0 & B_1 \cos \omega t \\ B_1 \cos \omega t & -B_0 \end{pmatrix} \]

Before we found that

\[ \psi(t) = \cos \left( \frac{w_1}{\hbar} t \right) | 0 \rangle + e^{i(w_0 + \pi)} \sin \left( \frac{w_1}{\hbar} t \right) | 1 \rangle \]

where $w_0 \propto B_0, w_1 \propto B_1, w_1 = \frac{\partial \theta}{\partial t}$

Now we can map our atomic system onto the spin problem, and we see that

\[ w_0 = \frac{E_1 - E_0}{\hbar} \rightarrow qubit energy splitting \]

\[ w_1 = \frac{V_1}{\hbar} \rightarrow \text{intensity of light at } w_0 \text{ shined on qubit} \]

$| 1 \rangle$ now changes latitude \[ \theta \] on Bloch sphere at rate $w_1 = \frac{V_1}{\hbar}$. Can now control very precisely.
Ok, so now we see that we can control
\[ |\psi\rangle = \alpha |0\rangle + \beta |1\rangle = \cos \frac{\theta}{2} |0\rangle + e^{i\phi} \sin \frac{\theta}{2} |1\rangle \]

Very precisely by shining well-timed pulses of light at the atomic qubit.

\[ |\psi\rangle = \alpha |0\rangle + \beta |1\rangle \]

\[ \Rightarrow \text{How do we measure } \alpha \text{ and } \beta ? \]

\[ \underline{Answer: Use fluorescence!} \]

Pick an atom that has a 3rd state, \( |2\rangle \), that couples to \( |1\rangle \) but not to \( |0\rangle \)!!

\[ \begin{align*}
|0\rangle, E_0 \\
|1\rangle, E_1 \\
|2\rangle, E_2
\end{align*} \]

\[ \Rightarrow \text{Shine light on atom at frequency } \omega = \frac{\Delta E}{h} = \frac{E_2 - E_1}{h} \]

\[ \Rightarrow \text{If atom is in state } |0\rangle \]

\[ \Rightarrow \text{NOTHING Happens (dark state)} \]

\[ \underline{BUT, if atom is in state } |1\rangle \Rightarrow \text{electron will absorb photon and get pushed up to state } |2\rangle \]

\[ \text{Electron will then tend to fall back down and re-radiate the photon. This can be detected!} \]
\[ |\psi\rangle = \alpha |10\rangle + \beta |11\rangle \]

**Measurement Scheme:**

\[ \begin{align*}
|12\rangle, E_2 & \\
-10, E, & \text{kw} = E_2 - E_1 \\
-10, E_0 & \\
\end{align*} \]

If electron in state \( |10\rangle \): No absorption

If electron in state \( |11\rangle \): Absorption + Re-radiation & detection

\[ \text{Detector \ kw} \]

\[ \xrightarrow{\text{This is JUST like Stern-Gerlach, in the sense that we get a DIFFERENT signal if electron is in state } |10\rangle \text{ or } |11\rangle} \]

Recipe for measuring \(|\alpha|, |\beta|\): Prepare atom in state \( |\psi\rangle = \alpha |10\rangle + \beta |11\rangle \) ⇒ Shine light on it ⇒ ask: did it absorb photon?

⇒ Prepare state again ⇒ shine light ⇒ did it absorb photon?

⇒ \( |11\rangle \)

Suppose you do this 1000 times and it absorbs the photon 800 times ⇒ What is \( \beta \)??

\[ |\beta|^2 = \text{probability} = \frac{800}{1000} = \frac{8}{10} \quad \Rightarrow \quad |\beta| \approx \sqrt{\frac{8}{10}} \]

\[ \Rightarrow |\alpha| = \sqrt{\frac{2}{10}} \]

But what about relative phase between \( \alpha \) and \( \beta \)? More difficult: Have to rotate \( |\psi\rangle \) by 90° around \( \hat{z} \) and measure \( \langle s_3 \rangle \) to get \( \langle s_3 \rangle = \langle s_2 \rangle \) ⇒ Rotate by 90° around \( \hat{s}_3 \) and measure \( \langle s_2 \rangle \) to get \( \langle s_2 \rangle = \langle s_3 \rangle \) \( (s_2, s_3) \) define \( \phi \)!! Trickier!