1 Photons

Photons are pretty versatile, and there are may ways to use photons as qubits! First, let’s think about what a photon is.

Consider classical electricity and magnetism, which are governed by Maxwell’s equations. Let’s have a look at Maxwell’s equations in an insulating, non-magnetic dielectric material:

\[
\begin{align*}
\nabla \cdot \varepsilon \vec{E} &= 0 \\
\n\nabla \times \vec{E} &= -\frac{1}{c} \frac{\partial \vec{B}}{\partial t} \\
\n\nabla \cdot \vec{B} &= 0 \\
\n\n\nabla \times \vec{B} &= \frac{\varepsilon}{c} \frac{\partial \vec{E}}{\partial t}
\end{align*}
\] (1)

For a dielectric material, \(\varepsilon\) is the dielectric constant which determines the polarizability of the material.

The great triumph of Maxwell’s equations is the prediction of traveling E and M waves. (The behavior of a photon is closely linked to the classical \(\vec{E}\) and \(\vec{B}\) fields.) A wave solution for the electric field might look like:

\[
\vec{E}(\vec{r},t) = \vec{E}_o \cos \left( \vec{k} \cdot \vec{r} - \omega t \right)
\] (2)

We also have that the magnetic field must be perpendicular to the electric field for a wave solution, so \(\vec{B}(\vec{r},t) = \vec{B}_o \cos \left( \vec{k} \cdot \vec{r} - \omega t \right)\), with \(\vec{B}_o\) perpendicular to \(\vec{E}_o\). Maxwell’s equations also force \(\vec{E}\) and \(\vec{B}\) to be perpendicular to the propagation vector \(\vec{k}\), so we are given a natural orthogonal basis set for 3D, with \(\vec{E}_o \times \vec{B}_o\) parallel to \(\vec{k}\).

In cgs units, \(|\vec{B}| = |\vec{E}|\), so let’s ignore the \(\vec{B}\)-field entirely since if we know \(\vec{E}\) then we know \(\vec{B}\). Also in a material it’s the \(\vec{E}\)-field that couples more strongly to electrons (\(\vec{F} = q\vec{E}\)), so the \(\vec{E}\)-field is more important.

So, we have traveling wave solutions. The phase velocity of any wave is given by \(v_{ph} = \frac{\omega}{k} = \frac{c}{\sqrt{\varepsilon}}\), where \(k = \frac{\omega}{c} \sqrt{\varepsilon}\). Perhaps more familiar is the index of refraction, \(n = \sqrt{\varepsilon}\). For light, we have \(v_{light} = \frac{c}{n}\), with \(\sqrt{\varepsilon} = 1\) for vacuum. \(\varepsilon\) depends on material properties, and relates to the polarizability of a material: \(\varepsilon = 1 + \frac{4\pi |\vec{p}|}{|\vec{E}|}\), where \(|\vec{p}| = \text{(dipole moment)/volume}\).

The more they polarize the bigger \(\varepsilon\) is and the slower light travels! Note: some materials might polarize more in one direction than another. The speed of light would thus be different for different polarizations of light.
Definition: The polarization of light is the direction that $\vec{E}$ points.

So what is a polarizer? This is a material that only allows light to pass with $\vec{E}$ polarized in a particular direction.

What about the energy of light? Classically a light wave fills a volume and has an energy density associated with it:

$$\rho = \frac{\text{energy}}{\text{volume}} = \frac{|\vec{E}|^2}{8\pi}$$  \hspace{1cm} (3)

So the energy of a light wave in a volume of space ($V$) is:

$$U = \rho V = \frac{|\vec{E}|^2}{8\pi} V$$  \hspace{1cm} (4)

Fact: Light actually travels in packets of energy called photons. Each photon has an $\vec{E}$-field and a frequency ($\omega$) associated with it. In quantum mechanics, we know that the energy of a particle is proportional to its frequency, so we should have that $U_{\text{photon}} = \hbar \omega$. This is the quanta of energy associated with a the particle of light, the photon.

So why is light quantized? There are different ways to approach this. The full treatment is known as quantum electro-dynamics. We will not derive this, but if you are interested there is an excellent book by Richard Feynman on the subject entitled *QED*.

One way to think about the quantization of light is that if I have a light in a cavity (i.e. a box), the space inside the cavity can be thought of as a bunch of simple harmonic oscillators with different frequencies. Each frequency is a mode of the cavity, and just like waves on a string (or particle in a box!!), we have a discrete spectrum of allowable modes which fit in the cavity:

Consequence of Quantization:

1. The number of photons ($N$) in a light wave depends on magnitude of $\vec{E}$-field:

$$U = \frac{E_0^2}{8\pi} V = N\hbar \omega \quad \Rightarrow \quad N = \frac{E_0^2 V}{8\pi \hbar \omega}$$  \hspace{1cm} (5)
2. Probabilistic behavior: Suppose we shine light on a polarizer. How do we interpret the behavior in terms of photons?

In region 1 we have \( N \) photons, all identical with \( \vec{E}_0 \) and \( \omega \). The number of photons in region 1 is given by:

\[
N = \frac{E_0^2 V}{8\pi\hbar\omega}.
\]

In region 2, we have \( N' \) photons, all identical with \( \vec{E}'_0 \) and \( \omega \). The number of photons in region 2 is given by:

\[
N' = \frac{E'_0^2 V}{8\pi\hbar\omega} = \frac{1}{2} N = \frac{E_0^2 V}{16\pi\hbar\omega}.
\]

We then conclude that

\[
\frac{N'}{N} = \frac{1}{2}.
\]

(6)

But this is strange since all the photons in region 1 are identical. Why do only half get through? Transmission must be probabilistic process, and we are therefore led to a probabilistic/QM interpretation. The probability of transmission for the preceding example is readily given by:

\[
prob = \frac{E_{ox}^2}{E_{ox}^2 + E_{oy}^2}.
\]

(7)

Thus the components of the \( \vec{E} \)-field in different directions act as probability amplitudes, just like the probability for measuring \( |0\rangle \) on a general state \( |\psi\rangle = \alpha|0\rangle + \beta|1\rangle \) is:

\[
prob = \frac{|\alpha|^2}{|\alpha|^2 + |\beta|^2}.
\]

(8)

So we can think of the components of an \( \vec{E} \)-field vector as a QM observable! The photon with electric field \( \vec{E} = E_{ox}\hat{x} + E_{oy}\hat{y} \) can be thought of as living in a state \( |\psi\rangle \) where:

\[
|\psi\rangle = \begin{pmatrix} \psi_x \\ \psi_y \end{pmatrix}
\]

with \( \psi_x (\psi_y) \) being the x-polarized (y-polarized) component.
The $\hat{x}$ and $\hat{y}$ polarization vectors form a basis! Suppose we have $\vec{E}_o = E_{ox}\hat{x} + E_{oy}\hat{y}$. We can rewrite this state in familiar QM language:

$$|\psi\rangle = \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix} = \cos \theta |x\rangle + \sin \theta |y\rangle$$  \hspace{1cm} (10)

The polarization of the photon is our new qubit variable: $|0\rangle = |x\rangle$ and $|1\rangle = |y\rangle$. Nice! Now how do we measure and transform this qubit?

Here we again draw an analogy to spin, and the polarizer plays the same role a Stern-Gerlach device.

Consider the Bloch sphere with general quantum state vector $|\psi\rangle = \cos \frac{\theta}{2} |0\rangle + e^{i\phi} \sin \frac{\theta}{2} |1\rangle$. A polarizer at angle $\theta'$ w.r.t. $\hat{x}$ leads to:

$$|\psi\rangle_{\text{photon}} = \cos \theta' |x\rangle + \sin \theta' |y\rangle = \cos \frac{\theta}{2} |0\rangle + e^{i\phi} \sin \frac{\theta}{2} |1\rangle$$  \hspace{1cm} (11)

This equality leads us to identify $\theta = 2\theta'$ and $\phi = 0$ in this case.

For this to be analogous to spins, however, we must be able to vary $\phi$. How do we do this? Is it even possible to get a relative phase terms with polarization?

The answer of course is yes, and the way this is done is to find a material where the index of refraction is different in $\hat{x}$ and $\hat{y}$ directions! This means that the velocity is different for $\hat{x}$ and $\hat{y}$ components. This is the case in anisotropic media with different polarizability in $\hat{x}$ and $\hat{y}$:
If this is the case, then light polarized along $\hat{y}$ will go $\approx 10\%$ faster than light polarized along the $\hat{x}$ direction. If the material has length $\Delta l$ then we can calculate the phase difference for light passing through to be $\Delta \phi = k \Delta l$:

For light with $\vec{E} || \hat{y}$, we have $k_y = \frac{\omega}{c} n_y$. Similarly, for light with $\vec{E} || \hat{x}$, we have $k_x = \frac{\omega}{c} n_x$. Thus the phase change through the material for the $y$-component (x-component) is $\Delta \phi_y = \frac{\omega}{c} n_y \Delta l$ ($\Delta \phi_x = \frac{\omega}{c} n_x \Delta l$). So, if our input state is $|\psi\rangle_{in} = \cos \frac{\theta}{2} |x\rangle + \sin \frac{\theta}{2} |y\rangle$, then the output state is:

$$|\psi\rangle_{out} = \cos \frac{\theta}{2} e^{i \frac{\pi}{2} n_y \Delta l} |x\rangle + \sin \frac{\theta}{2} e^{i \frac{\pi}{2} n_x \Delta l} |y\rangle$$

(12)

Rewriting, we see:

$$|\psi\rangle_{out} = \cos \frac{\theta}{2} |x\rangle + \sin \frac{\theta}{2} e^{i \frac{\pi}{2} (n_y - n_x) \Delta l} |y\rangle$$

(13)

so $\phi$ on the Bloch sphere is $\frac{\omega}{c} \Delta l (n_y - n_x)$. Therefore we see that this anisotropic medium has played the same role that the transverse $\vec{B}$-field did for spin.