

W.D.

Introduction to Regenerative Simulation, Lavenberg and Slutz

Simulation is a statistical experiment.

In addition ~~to~~ to measuring something, we would like to have an estimate of ~~on~~ the accuracy of the measured parameter.

The idea is to take multiple measurements of the same item and use standard statistical techniques to estimate confidence intervals.

Repetition methods

1. multiple simulation runs
2. dividing the run into approximately independent sub-runs and treating them as iid
3. analyzing the time ~~on~~ series of observations as a stationary time series (omitting initial transient) *study?*
4. regenerative simulation - finding regeneration points, and using the appropriate iid measurements.

regeneration point -system stochastically restarts

(e.g. start of busy period in queueing system)

We will ~~now~~ calculate the confidence intervals for a simulation of an M/G/1 queueing system.

let λ : arrival rate

service time T , k 'th moment β_k

$$(\beta_1 = \beta)$$

$$\beta_1, \beta_2 < \infty$$

$$\rho = \lambda \beta$$

g_K " queuing time for K th customer

w_K = waiting " " " " " (flow time)

Then $Q = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n g_k$ one queuing time

$W = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n w_k$ one waiting time (flow)

problem is to estimate Q (with conf, int)
arrival of customer to empty system
is regeneration point.

$t_k =$ ^{time of} k th regen point.

$(t_{K+1}, t_K) = k$ th four.

$v_k =$ # of wait served during k th four

$\sigma_k =$ sum of queuing times for all
wait served during k th four.

$v_k \sim v$ iid. series (is distributed as)

$\sigma_k \sim \sigma$ " "

estimate $Q = \lim_{n \rightarrow \infty} \frac{\sum_{k=1}^n \sigma_k}{\sum_{k=1}^n v_k} = \frac{E[\sigma]}{E[v]}$

using \hat{v}_k & $\hat{\sigma}_k$,
(confidentiality, σ, v unknown)

which

$\uparrow n \text{ limit}$ compare to $\frac{1}{n} \sum \frac{\sigma_k}{v_k}$
NOT SAME

estimate will be $\bar{\sigma}$

$$Q(n) = \sum_{k=1}^n \sigma_k / \sum_{k=1}^n v_k$$

consider $\sigma_k - Qv_k$

$$\begin{aligned} & E[(\sigma - Qv)^2] \\ &= E[\sigma^2] + Q^2 E[v^2] - 2QE[\sigma v] \\ &= \text{Var}(\sigma) + \bar{\sigma}^2 + Q^2 \text{Var}(v) \\ &+ Q^2 \bar{v}^2 - 2Q \text{Cov}(\sigma, v) \\ &\rightarrow 2Q \bar{v} \bar{\sigma} \\ \text{but } & + \bar{\sigma}^2 + Q^2 \bar{v}^2 - 2Q \bar{v} \bar{\sigma} \\ &= (\bar{\sigma} - Q\bar{v})^2 = 0 \end{aligned}$$

where $\text{Var}(x) = \frac{1}{n} \sum_{i=1}^n (x_i - \bar{x})^2 / n$

$$\therefore \sum_{i=1}^n (x_i - \bar{x})^2 p(x_i) = E((x - \bar{x})^2)$$

$$\text{Cov}(x, y) = \sum_{x_i} \sum_{y_i} (x_i - \bar{x})(y_i - \bar{y}) p(x_i, y_i)$$

$$\text{let } \sigma(n) = \sum_{k=1}^n \frac{\sigma_k}{n}$$

$$v(n) = \sum_{k=1}^n v_k / n$$

$$\Phi(t) = \left(\frac{1}{2\pi}\right)^{\frac{1}{2}} \int_{-\infty}^t e^{-x^2/2} dx \quad (\text{cumulative})$$

$$t \sim N(0, 1)$$

Then by central limit theorem

$$\lim_{n \rightarrow \infty} P_n \left\{ [\sigma(n) - Qv(n)] / (V(n)/n)^{\frac{1}{2}} \right\} < t \} = \phi(t)$$

(since if $\sigma_k - Qv_k = S_k$
 then $\sigma(n) - Qv(n) = \frac{\sum^n S_k}{n}$
 which has variance V/n , ($V = \frac{nV}{n^2}$)
 standard deviation $\sqrt{V/n}$
 ie $N(0, \frac{V}{n})$)

$$\text{let } V_1(n) = \sum_{k=1}^n (\sigma_k - \sigma(n))^2 / (n-1) = \hat{V}_{AR}(\sigma(n))$$

$$V_2(n) = \sum_{k=1}^n [v_k - v(n)]^2 / (n-1) = \hat{V}_{AR}(v(n))$$

$$V_{12} = \sum_{k=1}^n (\sigma_k - \sigma(n))(v_k - v(n)) / (n-1) \\ = \hat{\text{covar}}(v(n), \sigma(n))$$

~~16~~ and

$$V(n) = V_1(n) - 2Q(n)V_{12}(n) + [\hat{Q}(n)]^2 V_2(n)$$

ICBS $v(n) \rightarrow V$ as $n \rightarrow \infty$

~~opposite~~ ICOS $\lim_{n \rightarrow \infty} P_n \left\{ [\sigma(n) - Qv(n)] / (V(n)/n)^{\frac{1}{2}} \right\} < t \} = \phi(t)$
 (from central limit theor.)

$$\sigma(n) - Q \nu(n)$$

Consider

$$\frac{\sigma(n) - Q \nu(n)}{(\nu(n)/n)^{\frac{1}{2}}} < t$$

$$\frac{Q(n) - Q}{\sqrt{\frac{\nu(n)}{n}}} < \frac{t}{\nu(n)} \quad \text{since } Q(n) = \frac{\sigma(n)}{\nu(n)}$$

$$Q(n) - Q < \frac{t \sqrt{\nu(n)}}{\nu(n) \sqrt{n}}$$

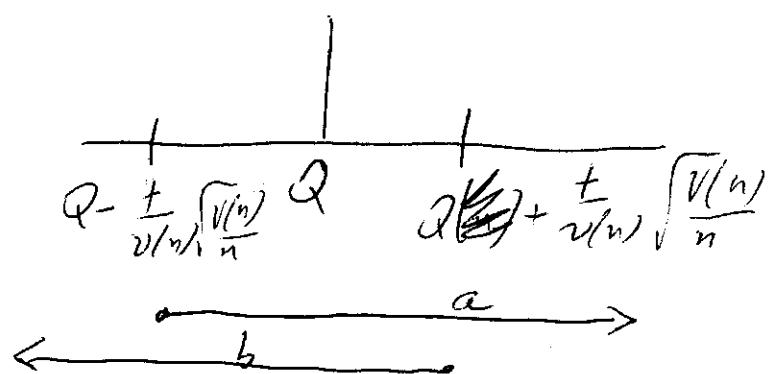
so

~~to follow~~

$$\Pr_n \left(\left[Q(n) - \frac{t}{\nu(n)} \sqrt{\frac{\nu(n)}{n}} \right] > Q \right) = \varphi(t)$$

and by symmetry

$$\Pr_n \left(\left(Q(n) + \frac{t}{\nu(n)} \sqrt{\frac{\nu(n)}{n}} \right) > Q \right) = \varphi(t)$$



$$P_2(Q(n) \in a) = \phi(t)$$

$$P_2(Q(n) \in b) = \phi(t)$$

$$P_2(Q(n) \in (a \cap b)) = 2\phi(t) - 1$$

So

$$\lim_{n \rightarrow \infty} P_2 \left\{ Q(n) - \frac{t + \sqrt{V(n)}}{\sqrt{n}} < Q < Q(n) + \frac{t - \sqrt{V(n)}}{\sqrt{n}} \right\} \\ = 2\phi(t) - 1$$

Thus if we want a confidence interval of $100(1-\alpha)\%$ for Q ,

$$\text{i.e. } 2\phi(t) - 1 = \alpha$$

$$t = \phi^{-1}\left(\frac{\alpha+1}{2}\right)$$

$$I_Q(n, \alpha) = [Q(n) - S(n, \alpha), Q(n) + S(n, \alpha)]$$

is a $(100-\alpha)\%$ conf interval
where

$$S(n, \alpha) = \phi^{-1}\left(\frac{\alpha+1}{2}\right) \frac{1}{\sqrt{n}} \sqrt{V(n)}$$

(in limit as $n \rightarrow \infty$)

note if $\alpha = .95$

$$\varphi^{-1}\left(\frac{1+\alpha}{2}\right) = 1.96 \approx 2$$

estimate for $w(n) = \bar{x}(n) + \beta$

with some conf interval.
 $(\beta \text{ known})$

But this holds only as $n \rightarrow \infty$.

for finite n , conf intervals
are wider (even in V)

e.g. m/G/1 $\rho = .8, \alpha = .95$
 $\rho = .12$

n	coverage(.8)	
50	.68	.83
100	.78	.81
250	.79	.91
500	.88	.93
1000	.87	.91

acceptable for $n > 500$ or so.
& for small ρ

Simulation Duration. $O(n/\alpha(1))$

ICBS $n(\alpha, \delta)$ = number of ~~customers~~^{fauls} required for confidence level α with width $\delta \cdot W$ grows as $\frac{1}{1-\rho} \quad (\rho \rightarrow 1)$

$E(v)n(\alpha, \delta)$ = # of customers grows as $\frac{1}{(1-\rho)^2} \quad (\rho \rightarrow 1)$

Comments

System must be regenerative

1st two moments of approp. variables must be $< \infty$