Introduction

**Motivation**

- Moving from rendering to simulation, animation
- Basic differential geometry crucial
  - How to compute frames, curvature, rotations
- This lecture relates to geometry, but focuses more on continuous concepts
- Future lectures deal with animation and simulation
- Quite mathematical, useful knowledge

**Outline**

- Parametric Curves
- Parametric Surfaces
- Rotations in 3D

**Parametric Curves (later Surfaces)**

- Curve is a geometric entity (set of points in space)
- Any local region is isomorphic to a line
- Generator function \( x(t) \)
  - Vector valued, or scalar function for each dimension.
  - Particular parameterization is arbitrary and not unique (not intrinsic to the curve)

**Arclength**

- Intrinsic parameterization of curve
  - \( x(s) = x(A^{-1}(s)) \)
    - But practical closed form may be hard to find
    - Unique up to sign change and translation

\[
\frac{dx(s)}{ds} = \frac{dx(t)}{dt} \left| \frac{dx(t)}{dt} \right|^{-1} \quad \text{and} \quad \left| \frac{dx(s)}{ds} \right| = 1
\]

**Tangents, Normals, Binormals**

- Tangent vector geometric property of curve
  - Intrinsic, independent of parameterization
  - Can exist where parametric velocity is 0 or undefined

\[
T = \frac{dx(s)}{ds}
\]
**Curvature and Normal**

Note: \( T \cdot T' = 1 \)

\[
(T \cdot T)' = (1)'
\]

Therefore: \( T \perp T' \)

We can write: \( T' = \kappa N \)

Curvature of the curve at this point

Normal of the curve at this point

Taylor expansion implies that if curvature is zero curve must be locally a straight line.

**Frenet Frame**

- Define binormal by \( B = T \times N \)
- Gives us an orthonormal coordinate frame
  - Moves along curve
  - Gives local frame of reference
  - Not defined at inflection points where no curvature
- Can find some nice demos online

**Osculating Plane**

- Defined by \( N \) and \( T \)
- Locally contains curve

**Normal Plane**

- Defined by \( N \) and \( B \)
- Locally perpendicular to curve

**Evolution of Frenet Frame**

\[
N' \perp N = N' = \alpha T + \beta B
\]

\[
\alpha = N' \cdot T
\]

\[
\beta = N' \cdot B
\]

Recall it’s an orthonormal basis.

Differentiate \( N \cdot T = 0 \) and \( N \cdot B = 0 \)

Yields \( N' \cdot T = -N \cdot \kappa N = -\kappa \)

\( N' \cdot B = -N \cdot (\tau)N = \tau \)

Therefore \( N' = -\kappa T + \tau B \)

We know \( T' = \kappa N \) and \( B' = -\tau B \)

**Torsion**

\[
B' = -\kappa N
\]

Change in binormal is then \( B' = -\kappa N \pm \tau B \)

If torsion is zero, we have a planar curve.

The minus sign is to make positive torsion CCW w.r.t. tangent.

**Evolution of Frenet Frame**

\[
T' = \kappa N
\]

\[
N' = -\kappa T + \tau B
\]

\[
B' = -\tau N
\]

ODE for evolution of Frenet Frame

Given starting point, if you know curvature and torsion, then you can build curve. (Need “speed” also if not arclength parameterized)

Discrete analogy: stacking up macaroni
**Radius of Curvature**

\[ x(s) = (r \cos \left( \frac{s}{r} \right), r \sin \left( \frac{s}{r} \right)) \]

Note that \( ||\mathbf{x}'|| = 1 \)

\[ T = [- \sin \left( \frac{s}{r} \right), \cos \left( \frac{s}{r} \right)] \]
\[ T' = [- \frac{1}{r} \cos \left( \frac{s}{r} \right), - \frac{1}{r} \sin \left( \frac{s}{r} \right)] \]
\[ \kappa = ||T'|| = \frac{1}{r} \]

Curvature is inverse of radius of curvature.

**Complicated Formulae**

For arclength parameterized curve

\[ \kappa = ||\mathbf{x}'(s)'|| \]
\[ \tau = \frac{\mathbf{x}'(s)' \times \mathbf{x}''(s)'}{||\mathbf{x}''(s)'||^2} \]

For arbitrarily parameterized curve

\[ \kappa = \frac{||\mathbf{x}'(t) \times \mathbf{x}''(t)||}{||\mathbf{x}'(t)||^3} \]
\[ \tau = \frac{\mathbf{x}'(t) \times \mathbf{x}''(t) \times \mathbf{x}'''(t)}{||\mathbf{x}'(t) \times \mathbf{x}''(t)||^2} \]

**Outline**

- Parametric Curves
- Parametric Surfaces
- Rotations in 3D

**Parametric Surfaces**

- Surface is geometric entity (set of points in space)
- Any local region is isomorphic to a plane
- Generator function \( \mathbf{x}(\mathbf{u}) \)
  - Vector valued, or scalar function for each dimension of embedding space (e.g. 2D surface embedded in 3D)
  - The parameter \( \mathbf{u} \) itself is of dimension two
  - Particular parameterization is arbitrary and not unique (not intrinsic to the surface)

**Tangent Space**

The tangent space at a point on a surface is the vector space spanned by:

\[ \frac{\partial \mathbf{x}(\mathbf{u})}{\partial u} \quad \frac{\partial \mathbf{x}(\mathbf{u})}{\partial v} \]

- Definition assumes that these directional derivatives are linearly independent.
- Tangent space of surface may exist even if the parameterization is bad.
- For surface the space is a plane.
- Generalized to higher dimension manifolds.

**Non-Orthogonal Tangents**

\[
\begin{align*}
\frac{\partial (x_1, x_2, x_3)}{\partial u} &= \left[ \cos(v/2) \cos(u/2) \right] \\
\frac{\partial (x_1, x_2, x_3)}{\partial v} &= \left[ \cos(\sqrt{2}v) \sin(u/2) \right]
\end{align*}
\]
Normals

- Normal at a point is unit vector perpendicular to the tangent space

\[ \mathbf{N} = \frac{\partial_u \mathbf{x} \times \partial_v \mathbf{x}}{||\partial_u \mathbf{x} \times \partial_v \mathbf{x}||} \]

First Fundamental Form

- Fundamental forms key concepts on surfaces

Pick a direction in parametric space: \( \mathbf{du} = [du, dv] \)

Corresponding direction in the tangent plane:

\[
\begin{align*}
\mathbf{dx} &= \frac{\partial \mathbf{x}}{\partial u} du + \frac{\partial \mathbf{x}}{\partial v} dv \\
\mathbf{dx} &= \nabla \mathbf{x}(u) \cdot \mathbf{du}
\end{align*}
\]

For unit speed in parametric space, the speed in the embedding space is

\[
\begin{align*}
\sigma^2 &= \mathbf{dx} \cdot \mathbf{dx} = \mathbf{du}^T \cdot (\nabla \mathbf{x}) \cdot (\nabla \mathbf{x})^T \cdot \mathbf{du} \\
\mathbf{dx} \cdot \mathbf{dx} &= \mathbf{du}^T \cdot \mathbf{I} \cdot \mathbf{du}
\end{align*}
\]

\[
I = \begin{bmatrix}
\partial_u \mathbf{x} \cdot \partial_u \mathbf{x} & \partial_u \mathbf{x} \cdot \partial_v \mathbf{x} \\
\partial_v \mathbf{x} \cdot \partial_u \mathbf{x} & \partial_v \mathbf{x} \cdot \partial_v \mathbf{x}
\end{bmatrix} = \begin{bmatrix}
I_{uu} & I_{uv} \\
I_{vu} & I_{vv}
\end{bmatrix}
\]

Properties of First Fundamental Form

- Encodes distance metric on surface
- For orthonormal tangents, simply identity
- Used as a metric by Green’s strain
- Invariant to translations and rotations

\[
\begin{align*}
(\partial x_u')(\partial x_v') &= (\partial x_u)(\partial x_v)(\partial x_u')(\partial x_v') \\
= R_{u,v}(\partial x_u)(\partial x_v) \\
\delta_{uv} &= \delta_{uu}(\partial x_u)(\partial x_u) \\
&= (\partial x_u)(\partial x_u)
\end{align*}
\]

ArcLength over Surface

\[
t = \int_a^b \left| \frac{dc(t)}{dt} \right| dt
\]

\[
= \int_a^b \sqrt{\mathbf{dx} \cdot \mathbf{dx}} dt
\]

\[
= \int_a^b \sqrt{\mathbf{du}^T \cdot \mathbf{I} \cdot \mathbf{du}} dt
\]

Principal Tangents

Bottom row is eigenvectors of \( I \)
Not intrinsic features of the surface!

Orthonormal Parameterization

Eigen decomposition of First Fundamental

\[
I = RS^2R^T = AA^T
\]

Define coordinate transform by

\[
\mathbf{du}' = SR^T \mathbf{du} = A^T \mathbf{du}
\]

\[
\mathbf{du} = R(1/S)\mathbf{du}' = A^{-T} \mathbf{du}'
\]

In transformed parameterization \( I \) is the identity:

\[
\mathbf{du}' \cdot I \cdot \mathbf{du}' = \mathbf{du}' \cdot \left(\frac{1}{S}\right) R^T \cdot \left( R S^T \right) R \cdot \left(\frac{1}{S}\right) \cdot \mathbf{du}'
\]

\[
= \mathbf{du}' \cdot \left(\frac{1}{S}\right) R^T \cdot R S^T \cdot R \cdot \left(\frac{1}{S}\right) \cdot \mathbf{du}'
\]

Similar to definition of arclength reparameterization.
Second Fundamental Form

Let \( dx \) be some tangent direction. \( dx = du \cdot \nabla x(u) \)

The directional derivative of the normal is:

\[
\nabla_u N = \frac{\partial N}{\partial u} du + \frac{\partial N}{\partial v} dv
\]

The normal is unit length so it is perpendicular to its derivative.

As shown in top-down view, the three vectors may not be co-planar. Surface may tilt to side as point moves.

Normal Curvature

\[
\kappa = \frac{du^T \cdot II \cdot du}{du^T \cdot I \cdot du}
\]

\[
\kappa = \frac{\partial u \cdot A^{-1} \cdot I \cdot A^{-T} \cdot du = du^T \cdot A^{-1} \cdot II \cdot A^{-T} \cdot du'}{\|du\|}
\]

\[
\kappa = \frac{du^T \cdot A^{-1} \cdot II \cdot A^{-T} \cdot du'}{\|du\|}
\]

Recall:

\[
1 = \text{isometry} = AA^T
\]

\[
du = du'(1/y) = A^{-1} du'
\]

Properties

\[
II = \begin{bmatrix}
-\frac{\partial u_x \cdot \partial u_N - \partial u_y \cdot \partial u_N}{\sqrt{g}} & \frac{\partial u_x \cdot \partial u_N - \partial u_y \cdot \partial u_N}{\sqrt{g}} \\
\frac{\partial u_y \cdot \partial u_N - \partial u_y \cdot \partial u_N}{\sqrt{g}} & \frac{\partial u_x \cdot \partial u_N - \partial u_y \cdot \partial u_N}{\sqrt{g}} 
\end{bmatrix}
\]

Symmetry

- Easy to show second version by expanding normal
- Box product with repeat is zero
- Any change in normal length will be perpendicular to surface
- Permutation of box product does not change results

Second Fundamental Form

\[
-T \cdot N_u = -du^T \begin{bmatrix}
-\frac{\partial u_x \cdot \partial u_N - \partial u_y \cdot \partial u_N}{\sqrt{g}} & -\frac{\partial u_x \cdot \partial u_N - \partial u_y \cdot \partial u_N}{\sqrt{g}} \\
-\frac{\partial u_y \cdot \partial u_N - \partial u_y \cdot \partial u_N}{\sqrt{g}} & -\frac{\partial u_x \cdot \partial u_N - \partial u_y \cdot \partial u_N}{\sqrt{g}} 
\end{bmatrix} du
\]

\[
= du^T \left[ -\frac{\partial u_x \cdot \partial u_N - \partial u_y \cdot \partial u_N}{\sqrt{g}} \right] du
\]

Matches definition of curvature for curve defined by cutting surface with the normal-tangent plane, but scaled by the surface metric.

Principal Curvatures

\[
\kappa = \frac{\text{du}^T \cdot II' \cdot \text{du'}}{\|\text{du}'\|}
\]

\[
II' = A^{-1} \cdot II \cdot A^{-T}
\]

Dot product projects away “twisting” curvature

- Eigenvectors are where there is nothing to project away
- Notice that it’s a real and symmetric matrix

\[
II' \cdot v = \kappa v
\]
Principal Curvatures

- Gaussian curvature $K = \kappa_1 \kappa_2$
- Mean curvature $H = \frac{\kappa_1 + \kappa_2}{2}$

Geodesic Curves

- Given a curve, $C(t)$ on a surface, $S$
- $C(t) = S(u(t), v(t))$
- The geodesic curvature is
  \[ \kappa_g = \kappa_u u' + \kappa_v v' \]
- Separates curvature into
  - What's necessary to stay on surface
  - What's wiggling in tangent plane
- Geodesics are curves with $\kappa_g = 0$
  - Generalize straight lines
  - Locally shortest path between points
  - On a cone they are great arcs

Outline

- Parametric Curves
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- Rotations in 3D

General 3D Rotations

- Non-commutative, much more complex than 2D
- General 3D axis, angle of rotation
- In axis-aligned case, simpler
  - In all cases, orthogonal matrices
  - Rows and columns of matrix are orthonormal

\[ R = \begin{bmatrix} \cos(\theta) & 0 & \sin(\theta) \\ 0 & 1 & 0 \\ \sin(\theta) & 0 & \cos(\theta) \end{bmatrix} \]
**Arbitrary 3D rotations**

Can be built from axis-aligned matrices:

$$\mathbf{R} = \mathbf{R}_z \cdot \mathbf{R}_y \cdot \mathbf{R}_x$$

Result due to Euler... hence called Euler Angles

Easy to store in vector

But NOT a vector:

$$\mathbf{R} = \text{rot}(x, y, z)$$

**Axis-Angle**

Direct representation of arbitrary rotation

AKA: axis-angle, angular displacement vector

Rotate $\theta$ degrees about some axis

Encode $\theta$ by length of vector

$$\theta = |\mathbf{r}|$$

**Axis Angle split Components**

Vector expressing a point has two parts

$\mathbf{x}_{\perp}$ does not change

$\mathbf{x}_{\perp}$ rotates like a 2D point

**Axis Angle Components**

**Rodriguez Formula**

$$\mathbf{x}' = \hat{\mathbf{r}}(\hat{\mathbf{r}} \cdot \mathbf{x}) + \sin(\theta)(\hat{\mathbf{r}} \times \mathbf{x}) - \cos(\theta)(\hat{\mathbf{r}} \times (\hat{\mathbf{r}} \times \mathbf{x}))$$

Linear in $\mathbf{x}$

Actually a minor variation...
**Rodriguez Matrix**

\[ x' = (\hat{r} \times) + \sin(\theta) \hat{r} \times - \cos(\theta) (\hat{r} \times)(\hat{r} \times) \hat{x} \]

\[
\begin{bmatrix}
0 & -\hat{r}_z & \hat{r}_y \\
\hat{r}_z & 0 & -\hat{r}_x \\
-\hat{r}_y & \hat{r}_x & 0
\end{bmatrix}
\]

Antisymmetric matrix

\[(a \times b) = (a \times b) \times \]

Easy to verify by expansion

**Exponential Maps**

- Allows tumbling
- No gimbal lock
- Orientations are space within \( \pi \) radius ball
- Nearly unique representations
- Singularities on shells at 2 \( \pi \)
- Nice for interpolation

**Exponentials: Basic Properties**

\[
e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots
\]

\[
e^{i0} = 1 + \frac{i\theta}{1!} + \frac{-\theta^2}{2!} + \frac{-i\theta^3}{3!} + \frac{\theta^4}{4!} + \cdots
\]

\[
= \left( 1 + \frac{-\theta^2}{2!} + \frac{\theta^4}{4!} + \cdots \right) + i \left( \frac{\theta}{1!} + \frac{-\theta^3}{3!} + \cdots \right)
\]

\[= \cos(\theta) + i \sin(\theta)\]

**Matrix Exponentials**

\[
e^{ix} = 1 + (\hat{r} \times)^0 + (\hat{r} \times)^0 + \frac{(\hat{r} \times)^0}{2!} + \frac{(\hat{r} \times)^0}{3!} + \frac{(\hat{r} \times)^0}{4!} + \cdots
\]

But notice that \( (\hat{r} \times)^3 = -(\hat{r} \times) \)

\[
e^{i\hat{r} \times \theta} = 1 + (\hat{r} \times)^0 \hat{r} \times + \frac{(\hat{r} \times)^0 (\hat{r} \times)^0 \hat{r} \times}{2!} + \frac{(\hat{r} \times)^0 (\hat{r} \times)^0 (\hat{r} \times)^0 \hat{r} \times}{3!} + \frac{(\hat{r} \times)^0 (\hat{r} \times)^0 (\hat{r} \times)^0 (\hat{r} \times)^0 \hat{r} \times}{4!} + \cdots
\]

\[
e^{(\hat{r} \times)^0} = (\hat{r} \times) \left( \frac{\theta}{1!} - \frac{\theta^3}{3!} + \cdots \right) + 1 + (\hat{r} \times)^2 \left( \frac{\theta^3}{2!} - \frac{\theta^4}{4!} + \cdots \right)
\]

\[
e^{(\hat{r} \times)^0} = (\hat{r} \times) \sin(\theta) + 1 + (\hat{r} \times)^2 (1 - \cos(\theta))
\]

**Quaternions**

- More popular than exponential maps
- Natural extension of complex numbers
- Hamilton 1843: interesting history
- Uber-complex numbers

\[
q = (z_1, z_2, z_3, s) = (\mathbf{z}, s)
\]

\[
\dot{q} = iz_1 + jz_2 + kz_3 + s
\]

\[
\dot{\hat{r}} = \hat{r} \times k^2 = -1
\]

\[
ij = k, ji = -k
\]

\[
jk = i, kj = -i
\]

\[
kj = j, jk = -j
\]

**Quaternion Properties**

Multiplication natural consequence of defn.

\[
q \cdot p = (z_q z_p + z_p z_q + z_p \times z_q, s_p s_q - z_p \cdot z_q)
\]

Conjugate

\[
q^* = (-z, s)
\]

Magnitude

\[
|q|^2 = \mathbf{z} \cdot \mathbf{z} + s^2 = q \cdot q^*
\]
Quaternion Rotations

Vectors as quaternions
\[ v = (v, 0) \]

Rotations as quaternions
\[ r = (\hat{r} \sin \frac{\theta}{2}, \cos \frac{\theta}{2}) \]

Rotating a vector
\[ x' = r \cdot x \cdot r^* \]

Composing rotations
\[ r = r_1 \cdot r_2 \quad \text{Compare to Exp. Map} \]

Quaternions

- No tumbling
- No gimbal lock
- Orientations are double unique
- Surface of unit 3-sphere in 4D
- Nice for interpolation
  - Slerps
  - Optimal quaternion splines

Interpolation

Ramamoorthi and Barr '97