Subdivision

- Very hot topic in computer graphics today
- Brief survey lecture, quickly discuss ideas
- Detailed study quite sophisticated
  - See some of materials from class webpage

Advantages
- Simple (only need subdivision rule)
- Local (only look at nearby vertices)
- Arbitrary topology (since only local)
- No seams (unlike joining spline patches)

Outline

- Basic Subdivision Schemes
- Analysis of Continuity
- Exact and Efficient Evaluation (Stam 98)

Subdivision Surfaces

- Coarse mesh & subdivision rule
  - Smooth surface = limit of sequence of refinements

Key Questions

- How to refine mesh?
- Where to place new vertices?
  - Provable properties about limit surface
Loop Subdivision Scheme

- How to refine mesh?
  - Refine each triangle into 4 triangles by splitting each edge and connecting new vertices

- Where to place new vertices?
  - Choose locations for new vertices as weighted average of original vertices in local neighborhood

Choose \( \beta \) by analyzing continuity of limit surface

- Original Loop
  \[ \beta = \frac{1}{n} \left( \frac{5}{8} - \frac{1}{4} \cos \frac{2\pi}{n} \right)^2 \]
- Warren
  \[ \beta = \begin{cases} \frac{5}{12} & n > 3 \\ \frac{3}{16} & n = 3 \end{cases} \]

Butterfly Subdivision

- Interpolating subdivision: larger neighborhood

Modified Butterfly Subdivision

- Need special weights near extraordinary vertices
  - For \( n = 3 \), weights are \( \frac{5}{12}, -\frac{1}{12}, -\frac{1}{12} \)
  - For \( n = 4 \), weights are \( \frac{3}{8}, 0, -\frac{1}{8}, 0 \)
  - For \( n \geq 5 \), weights are
    \[ \frac{1}{n} \left( \frac{1}{4} - \cos \frac{2\pi j}{n} \right) \left( \frac{1}{2} \cos \frac{4\pi j}{n} \right) \], \( j = 0 \ldots n - 1 \)
  - Weight of extraordinary vertex = 1 - \( \Sigma \) other weights
A Variety of Subdivision Schemes

- Triangles vs. Quads
- Interpolating vs. approximating

![Face split example](Zorin & Schröder)

More Exotic Methods

- Kobbelt’s subdivision:

![Kobbelt’s subdivision](Zorin & Schröder)

- Number of faces triples per iteration: gives finer control over polygon count

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Analyzing Subdivision Schemes

- Limit surface has provable smoothness properties

(From Zorin & Schröder)

### 4-Point Scheme

- What is the support?

  Step i:
  \[ v_2 \quad v_1 \quad v_0 \quad v_1 \quad v_2 \]

  Step i+1:
  \[ v_2 \quad v_1 \quad v_0 \quad v_1 \quad v_2 \]

  So, 5 new points depend on 5 old points

### Subdivision Matrix

- How are vertices in neighborhood refined?
  (with vertex renumbering like in last slide)

\[
\begin{pmatrix}
V^{(n+1)}_2 \\
V^{(n+1)}_1 \\
V^{(n+1)}_0 \\
V^{(n+1)}_1 \\
V^{(n+1)}_2 \\
\end{pmatrix} =
\begin{pmatrix}
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
\end{pmatrix}
\begin{pmatrix}
V^{(n)}_2 \\
V^{(n)}_1 \\
V^{(n)}_0 \\
V^{(n)}_1 \\
V^{(n)}_2 \\
\end{pmatrix}
\]

### Convergence Criterion

- Expand in eigenvectors of \( \mathbf{S} \):

\[
\bar{V}^{(n)} = \mathbf{S} \bar{V}^{(0)}
\]

\[
\mathbf{S} = \sum_{\lambda} \lambda \mathbf{e}_i
\]

\[
\bar{V}^{(n)} = \sum_{\lambda} \bar{V}^{(0)} \lambda \mathbf{e}_i
\]

**Criterion I:** \(|\lambda| \leq 1\)
Convergence Criterion

- What if all eigenvalues of $S$ are < 1?
  - All points converge to 0 with repeated subdivision

**Criterion II: $\lambda_0 = 1$**

Translation Invariance

- For any translation $t$, want:

\[
\begin{pmatrix}
v_{i+1}^{(n)} + t \\
v_{i}^{(n)} + t \\
v_{i-1}^{(n)} + t \\
v_{i+1}^{(n+1)} + t \\
v_{i}^{(n+1)} + t \\
v_{i-1}^{(n+1)} + t
\end{pmatrix}
= S
\begin{pmatrix}
v_{i+1}^{(n+1)} \\
v_{i}^{(n+1)} \\
v_{i-1}^{(n+1)} \\
v_{i+1}^{(n+2)} \\
v_{i}^{(n+2)} \\
v_{i-1}^{(n+2)}
\end{pmatrix} + t
\]

**Criterion III: $e_0 = 1$, all other $|\lambda_i| < 1$**

Smoothness Criterion

- Plug back in: $V^{(n+1)} = a_0 e_0 + \sum a_i \lambda_i e_i$
- Dominated by largest $\lambda_i$

**Case 1: $|\lambda_1| > |\lambda_2|$**

- Group of 5 points gets shorter
- All points approach multiples of $e_1 \rightarrow$ on a straight line
- Smooth!

**Case 2: $|\lambda_1| = |\lambda_2|$**

- Points can be anywhere in space spanned by $e_1, e_2$
- No longer have smoothness guarantee

**Criterion IV: Smooth iff $|\lambda_0| = 1 > |\mu_1| > |\mu_2|$**

Continuity and Smoothness

- So, what about 4-point scheme?
  - Eigenvalues = 1, 1/2, 1/4, 1/4, 1/8
  - $e_0 = 1$
  - Stable ✔
  - Translation invariant ✔
  - Smooth ✔

2-Point Scheme

- In contrast, consider 2-point interpolating scheme

\[
\begin{pmatrix}
\frac{1}{2} & \frac{1}{2} \\
0 & 1
\end{pmatrix}
\]

- Support = 3
- Subdivision matrix =

\[
\begin{pmatrix}
\frac{1}{2} & \frac{1}{2} & 0 \\
0 & 1 & 0 \\
0 & \frac{1}{2} & \frac{1}{2}
\end{pmatrix}
\]
Continuity of 2-Point Scheme

- Eigenvalues = 1, 1/2, 1/2
- $e_0 = 1$
- Stable $\surd$
- Translation invariant $\surd$
- Smooth $\surd$
  - Not smooth; in fact, this is piecewise linear

For Surfaces...

- Similar analysis: determine support, construct subdivision matrix, find eigenstuff
- Caveat 1: separate analysis for each vertex valence
- Caveat 2: consider more than 1 subdominant eigenvalue

Reif's smoothness condition: $\lambda_1 > |\lambda_2| \geq |\lambda_3| > |\lambda_i|$

- Points lie in subspace spanned by $e_1$ and $e_2$
  - If $|\lambda_1| \neq |\lambda_2|$, neighborhood stretched when subdivided, but remains 2-manifold

Fun with Subdivision Methods

Behavior of surfaces depends on eigenvalues

- Real
- Complex
- Degenerate

(recall that symmetric matrices have real eigenvalues)

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Practical Evaluation

- Problems with Uniform Subdivision
  - Exponential growth of control mesh
  - Need several subdivisions before error is small
  - Ok if you are "drawing and forgetting", otherwise ...
- (Exact) Evaluation at arbitrary points
- Tangent and other derivative evaluation needed
- Paper by Jos Stam SIGGRAPH 98 efficient method
  - Exact evaluation (essentially take out "subdivision")
  - Smoothness analysis methods used to evaluate

Isolated Extraordinary Points

- After 2+ subdivisions, isolated "extraordinary" points where irregular valence
- Regular region is usually easy
- For example, Catmull Clark can treat as B-Splines
Isolated Extraordinary Points

Subdivision Matrix

$C_1 = AC_0.$  
$C_n = AC_{n-1} = A^n C_0$

Eigen Space

$C_1 = AC_0.$  
$C_n = AC_{n-1} = A^n C_0$

$AV = V\Lambda \quad A = V\Lambda V^{-1}$

$C_n = AA^{n-1}C_0 = AVAV^{n-1}V^{-1}C_0$

$s_{\theta,\phi}(u,v) = C_0 \Theta \Phi^T b(u,v)$

$C_0 = V^{-1}C_0$

Only depends on valence of extraordinary vertex.

Comments

- Computing Eigen-Vectors is tricky
  - See Jos’ paper for details
  - He includes solutions for valence up to 500

- All eigenvalues are (abs) less than one
  - Except for lead value which is exactly one
  - Well defined limit behavior

- Exact evaluation allows “pushing to limit surface”

Curvature Plots

See Stam 98 for details
Summary

- Advantages:
  - Simple method for describing complex, smooth surfaces
  - Relatively easy to implement
  - Arbitrary topology
  - Local support
  - Guaranteed continuity
  - Multiresolution

- Difficulties:
  - Intuitive specification
  - Parameterization
  - Intersections

Pixar