Topics

• Vector and Tensor Fields
  • Divergence, curl, etc.

• Parametric Curves
  • Tangents, curvature, and etc.

• Parametric Surfaces
  • Normals, tangents, curvature, etc.

• Implicit Surfaces
  • Normals, curvature, etc.
Vectors

- A vector defines a magnitude and direction
  - Not just a list of numbers
    \[ \|\mathbf{v}\| \quad \hat{\mathbf{v}} = \frac{\mathbf{v}}{\|\mathbf{v}\|} \]
  - Particular numbers are an artifact of the coordinate system we chose
    - Not all coordinate systems are orthonormal
      \[ \mathbf{v} = [v_x, v_y, v_z] \]
  - Nearly everything that is useful can be defined w/o coordinate system
- Vectors transform like vectors
  \[ \mathbf{v}' = \mathbf{A} \cdot \mathbf{v} \]
- No set location (e.g. no root)
  - But may be functions of location
    \[ \mathbf{v} = \mathbf{v}(u) \]
    \[ \mathbf{v} = \mathbf{v}(x, y) \]
Tensors

- Tensors transform like tensors
- Tensors used to define oriented quantities
  - Independent of coordinate system
  - Specific realization will depend on coordinate system
    - Cartesian tensors -- orthonormal coordinate system
    - General tensors -- non-orthonormal coordinate system
- Tensors have rank
  - Not related to dimension of space
  - Rank 0 → scalars
  - Rank 1 → vectors
  - Rank 2 → matrices
  - Rank 3 → don’t work well in matrix-vector notation

\[ T' = A \cdot T \cdot A^T \]

\[ T' = A \cdot T \cdot A^{-1} \]
Tensors

- Examples

\[ \mathbf{a} \cdot \mathbf{b} = \mathbf{a}^T \cdot \mathbf{b} = ||\mathbf{a}|| \ ||\mathbf{b}|| \cos(\angle \mathbf{ab}) \]

\[ \mathbf{a} \cdot \mathbf{b}^T = \mathbf{P} \rightarrow \ (\mathbf{A} \cdot \mathbf{a}) \cdot (\mathbf{A} \cdot \mathbf{b})^T = \]
\[ \mathbf{A} \cdot (\mathbf{a} \cdot \mathbf{b}^T) \cdot \mathbf{A}^T = \mathbf{A} \cdot \mathbf{P} \cdot \mathbf{A}^T \]

\[ \mathbf{R} = \mathbf{x}' \cdot \mathbf{x}^T + \mathbf{y}' \cdot \mathbf{y}^T + \mathbf{z}' \cdot \mathbf{z}^T \]

\[ \mathbf{v}_1 \cdot \mathbf{v}_1^T + \mathbf{v}_2 \cdot \mathbf{v}_2^T = \mathbf{S} \]

\[ \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & \frac{3}{2} \\ \frac{3}{2} & 1 \end{pmatrix} \]

Note the way inner and outer products behave...
### Summation Notation

- **Notation due to Einstein**
  - Makes life much easier
  - Takes a while to get used to
  - Useful in other contexts as well

- **Free index**
  - Appears on both sides
  - Unique in each term
  - Implied “for all”

- **Dummy index**
  - Appears exactly twice in each term
  - Implied “sum over”

- **Different for general tensors**

\[
a \rightarrow a_i \quad A \rightarrow A_{ij}
\]
\[
s = a \cdot b \rightarrow s = a_i b_i
\]
\[
A = a \cdot b^T \rightarrow A_{ij} = a_i b_j
\]
\[
c = A \cdot b \rightarrow c_i = A_{ij} b_j
\]
\[
c^T = b^T \cdot A^T \rightarrow c_i = b_j A_{ij}
\]
\[
c = A \cdot b \rightarrow c_i = b_j A_{ij}
\]
\[
A' = RAR^T \rightarrow A'_{ij} = A_{kl} R_{ik} R_{jl}
\]
Summation Notation

• Two special symbols
  • Delta \( \delta_{ij} \)
  • Permutation \( \varepsilon_{ijk} \)

\[
\delta_{ij} = \begin{cases} 
1 & \text{if } i = j \\
0 & \text{if } i \neq j 
\end{cases}
\]

\[
\varepsilon_{ijk} = \begin{cases} 
1 & \text{if } i, j, k \text{ are even permutation of } 1, 2, 3 \\
-1 & \text{if } i, j, k \text{ are odd permutation of } 1, 2, 3 \\
0 & \text{else}
\end{cases}
\]

If you’re slumming in \( \mathbb{R}^2 \)

\[
\varepsilon_{ij} = \begin{cases} 
1 & \text{if } i, j \text{ are } 1, 2 \\
-1 & \text{if } i, j \text{ are } 2, 1 \\
0 & \text{else}
\end{cases}
\]
Scalar Fields

• Scalar as function of some spatial variable(s)
  
  • e.g.:
  
  \[ f(x, y) = f(x) = \sin(x) \sin(y) \]
Vector Fields

- Vector as function of some spatial variable(s)
  - e.g.:
    \[ \mathbf{v}(x, y) = \mathbf{v}(x) = [\sin(x), \cos(y)] \]
    \[ \mathbf{v}(x, y) = \mathbf{v}(x) = [1, \sin(y), 0] \]
Differential Operators on Fields

- Derivatives of field w.r.t. spatial coordinates
  - Coordinates implicit given field parameterization
  - Linear operators on the field
  - Not tied to any particular coordinate system

- Basic operators
  - Gradient
  - Divergence
  - Curl
  - Laplacian

- All expressed with $\nabla$ (a.k.a. Nabla or del)

\[
\nabla = \sum_i e_i \frac{\partial}{\partial x_i}
\]

\[
\nabla = \left[ \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right]
\]

\[
\nabla_i = \partial_i = \frac{\partial}{\partial x_i}
\]

Monday, October 26, 2009
Gradient

• Often applied to scalar fields
  • Gives direction of steepest accent
• Also has meaning for higher rank fields
  • Elevates rank by one
  • e.g. velocity gradient of a Newtonian fluid gives the strain rate

\[
\text{grad} f(\mathbf{x}) = \nabla f(\mathbf{x}) = \left[ \frac{\partial f(\mathbf{x})}{\partial x_1}, \frac{\partial f(\mathbf{x})}{\partial x_2}, \frac{\partial f(\mathbf{x})}{\partial x_3} \right],
\]

\[
f(\mathbf{x}) = x^2 + y^2
\]

\[
\nabla f(\mathbf{x}) = [2x, 2y]
\]
Divergence

• For a vector field it describes the net expansion or contraction
• Lowers rank by one
  • Divergence of vector field is a scalar
  • An inner product of derivatives with the field

\[
\text{div } \mathbf{v}(x) = \nabla \cdot \mathbf{v}(x) = \nabla^T \cdot \mathbf{v}(x) = \frac{\partial v_x(x)}{\partial x_1} + \frac{\partial v_y(x)}{\partial x_2} + \frac{\partial v_z(x)}{\partial x_3}
\]

\[
\nabla \cdot [\sin(x), \cos(y)] = -\cos(x) + \sin(y)
\]
Curl

- For a vector field it describes the net “rotation”
- Cross product of derivatives with the field
  - Scaler in 2D, vector in 3D

\[ \text{curl } \mathbf{v}(\mathbf{x}) = \nabla \times \mathbf{v}(\mathbf{x}) \]

\[ \nabla \times [\cos(y), 0] = -\sin(y) \]
Laplacian

- Divergence of Gradient
- Scalar second derivative operator
- Difference between a point and its surround
  - Often used for smoothing of some sort

\[ \nabla \cdot \nabla = \nabla^2 = \frac{\partial^2}{\partial xx} + \frac{\partial^2}{\partial yy} + \frac{\partial^2}{\partial zz} \]

\[ \cos^2(x) \sin^2(y) \]

\[ 2 \cos^2(x) \cos^2(y) - 4 \cos^2(x) \sin^2(y) + 2 \sin^2(x) \sin^2(y) \]
Notation Examples

\[ \mathbf{v}(\mathbf{x}) = \nabla f(\mathbf{x}) \quad \longrightarrow \quad v_i = \partial_i f \]
\[ s(\mathbf{x}) = \nabla \cdot \mathbf{v}(\mathbf{x}) \quad \longrightarrow \quad s = \partial_i v_i \]
\[ \mathbf{c}(\mathbf{x}) = \nabla \times \mathbf{v}(\mathbf{x}) \quad \longrightarrow \quad c_i = \varepsilon_{ijk} \partial_j v_k \]
\[ \mathbf{a}(\mathbf{x}) = (\mathbf{v}(\mathbf{x}) \cdot \nabla) \mathbf{b}(\mathbf{x}) \quad \longrightarrow \quad a_i = v_j \partial_j b_i \]
Fun Facts

\[ \nabla \cdot (\nabla \times \mathbf{v}) = 0 \]
\[ \nabla \times (\nabla s) = 0 \]

Both are obvious in tensor notation

- Helmholtz-Hodge decomposition
  - Smooth, differentiable vector field

\[ \mathbf{a} = \nabla s + \nabla \times \mathbf{v} + \mathbf{h} \]

- \( \nabla s \): irrotational or curl-free part
- \( \nabla \times \mathbf{v} \): solenoidal or divergence-free part
- \( \mathbf{h} \): harmonic part

Scalar and vector potentials
Directional Derivative

\[
\frac{df}{dx} = \mathbf{x} \cdot \nabla f
\]

Add a picture or something...
Parametric Curves

• Curve is a geometric entity
  • Set of points in space
  • In neighborhood of any point it is isomorphic to a line

• Generator function: \( \mathbf{x} = \mathbf{x}(t) \)
  • A vector valued function (careful with “vector”)
  • A scalar function for each dimension of embedding space

• A particular parameterization is arbitrary and not unique
  • Parameterization is not intrinsic

\[
[x(t)] = \begin{bmatrix}
\cos(\theta), \\
\sin(\theta)
\end{bmatrix}
\]

\[
\begin{bmatrix}
\frac{2u}{u^2 + 1}
\end{bmatrix}
\begin{bmatrix}
1 - u^2 \\
u^2 + 1
\end{bmatrix}
\]
Derivatives

• Given function for curve we can take derivatives w.r.t. the parameter:
\[ \dot{x} = \frac{dx}{dt} \]

• The derivatives have names based on physical analogs
  • Velocity
  • Acceleration
  • Jerk
  • Snap, Crackle, and Pop

• Speed is the magnitude of velocity \( s = ||\dot{x}|| \)

• All are dependent on parameterization and not intrinsic

• Note that, e.g., velocity is a vector field on \( t \)
Arclength

- Let $s = A(t) = \int_0^t \| x(\tau) \| \, d\tau$
- $A(t)$ is the arclength of the curve
- The arclength reparameterization of the curve is $\hat{x}(s) = x(A^{-1}(s))$
- The arclength parameterization is unique up to sign change and translation

\[
\frac{d\hat{x}(s)}{ds} = \frac{dx(t)}{dt} \left\| \frac{dx(t)}{dt} \right\|^{-1}
\]
and
\[
\left\| \frac{d\hat{x}(s)}{ds} \right\| = 1
\]

Closed form arclength parameterization may be hard to find.
Tangent Vector

- Tangent vector is a geometric property of the curve
  - Does not depend on parameterization
  - Tangent may exist where velocity is zero or may be undefined

\[
T = \frac{d\hat{x}(s)}{ds}
\]
Curvature and Normal

Note: \( \mathbf{T} \cdot \mathbf{T} = 1 \)
\[
(T \cdot T)' = (1)'
\]
\( \mathbf{T} \cdot \mathbf{T}' = 0 \)

Therefore: \( \mathbf{T} \perp \mathbf{T}' \)

We can write: \( \mathbf{T}' = \kappa \mathbf{N} \)

Curvature of the curve at this point
Normal of the curve at this point

Taylor expansion implies that if curvature is zero curve must be locally a straight line.
Frenet Frame

• Define *binormal* by \( \mathbf{B} = \mathbf{T} \times \mathbf{N} \)

• Gives us orthonormal coordinate frame: Frenet Frame
  • Moves along curve
  • Give local frame of reference

Not defined at inflection points where there is no curvature...
Frenet Frame

- Osculating Plane
  - Defined by N and T
  - Locally contains the curve

- Normal Plane
  - Defined by N and B
  - Locally perpendicular to the curve
Torsion

\[ B \cdot B = 1 \quad \rightarrow \quad B \cdot B' = 0 \]

\[ B \cdot T = 0 \quad \rightarrow \quad B' \cdot T + B \cdot T' = 0 \]
\[ \quad \rightarrow \quad B' \cdot T = -B \cdot T' = -B \cdot \kappa N = 0 \]

\[ B' \perp B \quad \text{and} \quad B' \perp T \]

Change in binormal is then \( B' = -\tau N \)

If torsion is zero, we have a planar curve.

The minus sign is to make positive torsion CCW w.r.t. tangent.
Evolution of Frenet Frame

\[ \mathbf{N}' \perp \mathbf{N} \rightarrow \mathbf{N}' = \alpha \mathbf{T} + \beta \mathbf{B} \]

\[ \alpha = \mathbf{N}' \cdot \mathbf{T} \]
\[ \beta = \mathbf{N}' \cdot \mathbf{B} \]

Recall it’s an orthonormal basis.

Differentiate \( \mathbf{N} \cdot \mathbf{T} = 0 \) and \( \mathbf{N} \cdot \mathbf{B} = 0 \)

Yields

\[ \mathbf{N}' \cdot \mathbf{T} = -\mathbf{N} \cdot \kappa \mathbf{N} = -\kappa \]
\[ \mathbf{N}' \cdot \mathbf{B} = -\mathbf{N} \cdot (-\tau) \mathbf{N} = \tau \]

Therefore

\[ \mathbf{N}' = -\kappa \mathbf{T} + \tau \mathbf{B} \]

We know \( \mathbf{T}' = \kappa \mathbf{N} \) and \( \mathbf{B}' = -\tau \mathbf{B} \)
Evolution of Frenet Frame

Given starting point, if you know curvature and torsion, then you can build curve.
(Need “speed” also if not arclength parameterized.)

Discrete analogy: stacking up macaroni

\[
\begin{align*}
T' &= \kappa N \\
N' &= -\kappa T + \tau B \\
B' &= -\tau N
\end{align*}
\]

\[
\begin{bmatrix}
T' \\
N' \\
B'
\end{bmatrix} =
\begin{bmatrix}
0 & \kappa & 0 \\
-\kappa & 0 & \tau \\
0 & -\tau & 0
\end{bmatrix}
\cdot
\begin{bmatrix}
T \\
N \\
B
\end{bmatrix}
\]
Radius of Curvature

\[ \mathbf{x}(s) = \left[ r \cos \left( \frac{s}{r} \right), r \sin \left( \frac{s}{r} \right) \right] \]

Note that \( \|\mathbf{x}'\| = 1 \)

\[ \mathbf{T} = \left[ -\sin \left( \frac{s}{r} \right), \cos \left( \frac{s}{r} \right) \right] \]

\[ \mathbf{T}' = \left[ -\frac{1}{r} \cos \left( \frac{s}{r} \right), -\frac{1}{r} \sin \left( \frac{s}{r} \right) \right] \]

\[ \kappa = \|\mathbf{T}'\| = \frac{1}{r} \]

Curvature is inverse of radius of curvature.
Some Formulae

• For arclength parameterized curve

\[ \kappa = \left\| \mathbf{x}''(s) \right\| \]
\[ \tau = \frac{\mathbf{x}' \cdot (\mathbf{x}'' \times \mathbf{x}''')}{{\left\| \mathbf{x}'' \right\|^2}} \]

• For arbitrarily parameterized curve

\[ \kappa = \frac{\left\| \mathbf{x}'(t) \times \mathbf{x}''(t) \right\|}{{\left\| \mathbf{x}'(t) \right\|^3}} \]
\[ \tau = \frac{\mathbf{x}'(t) \times \mathbf{x}''(t) \cdot \mathbf{x}'''(t)}{{\left\| \mathbf{x}'(t) \times \mathbf{x}''(t) \right\|^2}} \]
Field Evaluated Along a Curve

- Curve defined in some space
  - $x(t)$
- Function on embedding space of curve
  - $f(x)$
- Composition function
  - $f(x(t))$

\[
\frac{df}{dt} = \nabla f \cdot \frac{dx}{dt}
\]
Parametric Surfaces

- Surface is a geometric entity
  - Set of points in space
  - In neighborhood of any point it is isomorphic to a plane

- Generator function: $\mathbf{x}(\mathbf{u})$
  - A vector valued function (careful with “vector”)
  - A scalar function for each dimension of embedding space
  - Dimension of parameter is two

- A particular parameterization is arbitrary and not unique
  - Parameterization is not intrinsic
Derivatives

• Given function for curve we can take derivatives w.r.t. the parameter:
  \[ \frac{\partial x(u)}{\partial u} \quad \frac{\partial x(u)}{\partial v} \]

• All are dependent on parameterization and not intrinsic

• Note that each one is a vector field on \( u \)

• Examples of degeneracies

\[
[v^3, u, v^2] \quad [v^3, u, -v^5] \quad [v(u + 1), u(1 - v), 0]
\]
Tangent Space

• The *tangent space* at a point on a surface is the vector space spanned by

\[
\begin{align*}
\frac{\partial x(u)}{\partial u} & \quad \frac{\partial x(u)}{\partial v}
\end{align*}
\]

• Definition assumes that these directional derivatives are linearly independent.
• Tangent space of surface may exist even if the parameterization is bad

• For surface the space is a plane
  • Generalized to higher dimension manifolds
Non Orthogonal Tangents

\[
\begin{bmatrix}
\cos(\theta 2\pi) \cos(\phi \pi/2) \\
\sin(\theta 2\pi) \cos(\phi \pi/2) \\
\sin(\phi \pi/2)
\end{bmatrix}
\begin{bmatrix}
\cos(2\pi \theta) \cos \left( \frac{1}{2} \pi \left( \frac{1}{2} (1 - |\phi|) \cos(6\pi \theta) \phi + \phi \right) \right) \\
\cos \left( \frac{1}{2} \pi \left( \frac{1}{2} (1 - |\phi|) \cos(6\pi \theta) \phi + \phi \right) \right) \sin(2\pi \theta) \\
\sin \left( \frac{1}{2} \pi \left( \frac{1}{2} (1 - |\phi|) \cos(6\pi \theta) \phi + \phi \right) \right)
\end{bmatrix}
\]

\[\theta \in [0..1], \quad \phi \in [-1..1]\]
Normals

• The normal at a point is the unit vector perpendicular to the tangent space

\[ \mathbf{N} = \frac{\partial_u \mathbf{x} \times \partial_v \mathbf{x}}{||\partial_u \mathbf{x} \times \partial_v \mathbf{x}||} \]

• The normal direction is determined
  • Up to a sign change
  • Relative to surface
First Fundamental

Pick a direction in parametric space: \( \mathbf{du} = [d\mathbf{u}, d\mathbf{v}] \)

Corresponding direction in the tangent plane:
\[
d\mathbf{x} = \frac{\partial \mathbf{x}}{\partial \mathbf{u}} d\mathbf{u} + \frac{\partial \mathbf{x}}{\partial \mathbf{v}} d\mathbf{v}
\]
\[
d\mathbf{x} = \mathbf{du} \cdot \nabla \mathbf{x}(\mathbf{u})
\]

For unit speed in parametric space, the speed in the embedding space is
\[
s^2 = d\mathbf{x} \cdot d\mathbf{x} = \mathbf{du}^\mathsf{T} \cdot (\nabla \mathbf{x}) \cdot (\nabla \mathbf{x})^\mathsf{T} \cdot d\mathbf{u}
\]
\[
d\mathbf{x} \cdot d\mathbf{x} = \mathbf{du}^\mathsf{T} \cdot \mathbf{I} \cdot d\mathbf{u}
\]

\[\mathbf{I} = \begin{bmatrix}
\partial_u \mathbf{x} \cdot \partial_u \mathbf{x} & \partial_v \mathbf{x} \cdot \partial_u \mathbf{x} \\
\partial_u \mathbf{x} \cdot \partial_v \mathbf{x} & \partial_v \mathbf{x} \cdot \partial_v \mathbf{x}
\end{bmatrix}
\]
\[
I_{ij} = (\partial_i x_k)(\partial_j x_k)
\]
First Fundamental

\[ I = \begin{bmatrix} \partial_u x \cdot \partial_u x & \partial_v x \cdot \partial_u x \\ \partial_u x \cdot \partial_v x & \partial_v x \cdot \partial_v x \end{bmatrix} \quad I_{ij} = (\partial_i x_k)(\partial_j x_k) \]

- Encodes distance metric on the surface
- If tangents are orthonormal it reduces to identity
- Used as metric by Green’s Strain
- Invariant w.r.t. translations and rotations of surface:

\[ (\partial_i x_k')(\partial_j x_k') = (\partial_i R_{kp} x_p)(\partial_j R_{kq} x_q) \]

\[ = R_{kp} R_{kq} (\partial_i x_p)(\partial_j x_q) \]

\[ = \delta_{pq} (\partial_i x_p)(\partial_j x_q) \]

\[ = (\partial_i x_p)(\partial_j x_p) \]

e.g. \[ x_i' = R_{ij} x_j \]
First Fundamental

\[ I = \begin{bmatrix} \partial_x \mathbf{x} \cdot \partial_u \mathbf{x} & \partial_x \mathbf{x} \cdot \partial_u \mathbf{x} \\ \partial_x \mathbf{x} \cdot \partial_v \mathbf{x} & \partial_x \mathbf{x} \cdot \partial_v \mathbf{x} \end{bmatrix} \]

\[ I_{ij} = (\partial_i x_k)(\partial_j x_k) \]

- Transforms like a tensor in parameter space:

\[ u'_i = R_{ij} u_j \quad \longrightarrow \quad u_i = R_{ji} u'_j \]

Assume orthonormal transform...

\[ \frac{\partial x_k}{\partial u'_i} \frac{\partial x_k}{\partial u'_j} = \frac{\partial x_k}{\partial u_p} \frac{\partial u_p}{\partial u'_i} \frac{\partial x_k}{\partial u_q} \frac{\partial u_q}{\partial u'_j} \]

\[ = R_{ip} \frac{\partial x_k}{\partial u_p} \frac{\partial x_k}{\partial u_q} R_{jq} \]

\[ I'_{ij} = R_{ip} I_{pq} R_{jq} \]
Arclength Over Surface

\[ c(t) = x(u(t)) \]

\[ l = \int_a^b \left\| \frac{dc(t)}{dt} \right\| \, dt \]

\[ = \int_a^b \sqrt{\|dx\|^2} \, dt \]

\[ = \int_a^b \sqrt{dx \cdot dx} \, dt \]

\[ = \int_a^b \sqrt{du^T \cdot I \cdot du} \, dt \]
Principle Tangents

Bottom row is eigenvectors of $I$
Not intrinsic features of the surface!
Principle Tangents
Orthonormal Parameterization

Eigen decomposition of First Fundamental
\[ I = RS^2R^T = AA^T \]

Define coordinate transform by
\[ du' = SR^Tdu = A^Tdu \]
\[ du = R(1/S)du' = A^{-T}du' \]

In transformed parameterization \( I \) is the identity.
\[ du'^T \cdot I' \cdot du' = du'^T \cdot (1/S) \cdot R^T \cdot (R \cdot S^2 \cdot R^T) \cdot R \cdot (1/S) \cdot du' \]
\[ = du'^T \cdot \left( (1/S) \cdot R^T \cdot R \cdot S^2 \cdot R^T \cdot R \cdot (1/S) \right) du' \]

Similar to definition of arclength reparameterization.
Second Fundamental

Let \( dx \) be some tangent direction \( dx = du \cdot \nabla x(u) \)

The directional derivative of the normal is

\[
\nabla u \mathbf{N} = \frac{\partial \mathbf{N}}{\partial u} du + \frac{\partial \mathbf{N}}{\partial v} dv
\]

The normal is unit length so it is perpendicular to its derivative.

As shown in top-down view, the three vectors may not be co-planar. Surface may tilt to side as point moves.
Second Fundamental

Let \( \mathbf{dx} \) be some tangent direction \( \mathbf{dx} = \mathbf{du} \cdot \nabla x(u) \)

The directional derivative of the normal is

\[
\nabla u \mathbf{N} = \frac{\partial \mathbf{N}}{\partial u} du + \frac{\partial \mathbf{N}}{\partial v} dv
\]

The change in normal restricted to the plane containing the tangent and normal is given by

\[
-\mathbf{T} \cdot \mathbf{N}_T = -\mathbf{dx} \cdot \nabla u \mathbf{N}
\]

\[
= -(\mathbf{du} \cdot \nabla x) \cdot (\mathbf{du} \cdot \nabla \mathbf{N})
\]

\[
= \mathbf{du}^T \left[ \begin{array}{cc}
-\partial_u x \cdot \partial_u \mathbf{N} & -\partial_u x \cdot \partial_v \mathbf{N} \\
-\partial_v x \cdot \partial_u \mathbf{N} & -\partial_v x \cdot \partial_v \mathbf{N}
\end{array} \right] \mathbf{du}
\]
-\mathbf{T} \cdot \mathbf{N}_T = \mathbf{du}^T \left[ \begin{array}{cc} -\partial_u \mathbf{x} \cdot \partial_u \mathbf{N} & -\partial_u \mathbf{x} \cdot \partial_v \mathbf{N} \\ -\partial_v \mathbf{x} \cdot \partial_u \mathbf{N} & -\partial_v \mathbf{x} \cdot \partial_v \mathbf{N} \end{array} \right] \mathbf{du} \\
= \mathbf{du}^T \mathbf{II} \mathbf{du}

Matches definition of curvature for curve defined by cutting surface with the normal-tangent plane, but scaled by the surface metric.
Second Fundamental

\[ \mathbf{II} = \begin{bmatrix} -\partial_u x \cdot \partial_u N & -\partial_u x \cdot \partial_v N \\ -\partial_v x \cdot \partial_u N & -\partial_v x \cdot \partial_v N \\ \partial_{uu} x \cdot N & \partial_{uv} x \cdot N \\ \partial_{vu} x \cdot N & \partial_{vv} x \cdot N \end{bmatrix} \]

Symmetry

- Easy to show second version by expanding normal
  - Box product with repeat is zero
  - Any change in normal length will be perpendicular to surface
  - Permutation of box product does not change results
Osculating Paraboloid

• Tangent plane is linear approximation to surface at a point
• Osculating paraboloid is quadratic approximation to surface at a point
  • Matches surface’s First and Second Fundamentals at the point

\[ P(u) = c_0 + c_1u + c_2v + c_3u^2 + c_4uv + c_5v^2 \]
Nature of Surface

\[ P(u) = c_0 + c_1 u + c_2 v + c_3 u^2 + c_4 uv + c_5 v^2 \]

**Elliptic**

\[ c_3 c_4 - (c_5/2)^2 > 0 \]

**Hyperbolic**

\[ c_3 c_4 - (c_5/2)^2 < 0 \]

**Parabolic**

\[ c_3 c_4 - (c_5/2)^2 = 0 \]

Includes planar case
Normal Curvature

• Curvature adjusted for surface metric and for velocity in parameter space:

\[ \kappa = \frac{\mathbf{d}u^T \cdot \mathbf{II} \cdot \mathbf{d}u}{\mathbf{d}u^T \cdot \mathbf{I} \cdot \mathbf{d}u} \]
Normal Curvature

\[ \kappa = \frac{\mathbf{d}u^T \cdot \mathbf{II} \cdot \mathbf{d}u}{\mathbf{d}u^T \cdot \mathbf{I} \cdot \mathbf{d}u} \]

\[ \kappa \mathbf{d}u^T \cdot \mathbf{I} \cdot \mathbf{d}u = \mathbf{d}u^T \cdot \mathbf{II} \cdot \mathbf{d}u \]

\[ \kappa \mathbf{d}u'^T \cdot \mathbf{A}^{-1} \cdot \mathbf{I} \cdot \mathbf{A}^{-T} \cdot \mathbf{d}u' = \mathbf{d}u'^T \cdot \mathbf{A}^{-1} \cdot \mathbf{II} \cdot \mathbf{A}^{-T} \cdot \mathbf{d}u' \]

\[ \kappa \mathbf{d}u'^T \cdot \mathbf{d}u' = \mathbf{d}u'^T \cdot \mathbf{A}^{-1} \cdot \mathbf{II} \cdot \mathbf{A}^{-T} \cdot \mathbf{d}u' \]

Recall

\[ \mathbf{I} = \mathbf{R}S^2\mathbf{R}^T = \mathbf{A}\mathbf{A}^T \]

\[ \mathbf{d}u = \mathbf{R}(1/S)\mathbf{d}u' = \mathbf{A}^{-T}\mathbf{d}u' \]
Principal Curvatures

\[ \kappa = \frac{\mathbf{du}'^T \cdot \mathbf{II}' \cdot \mathbf{du}'}{||\mathbf{du}'||} \quad \mathbf{II}' = \mathbf{A}^{-1} \cdot \mathbf{II} \cdot \mathbf{A}^{-T} \]

- Dot product projects away “twisting” curvature
- Eigenvectors are where there is nothing to project away
  - Notice that it’s a real and symmetric matrix

\[ \mathbf{II}' \cdot \mathbf{v} = \kappa \mathbf{v} \]
Principal Curvatures

\[ \Pi' \cdot \mathbf{v} = \kappa \mathbf{v} \]

Elliptic \[ \kappa_1 \kappa_2 > 0 \]

Hyperbolic \[ \kappa_1 \kappa_2 < 0 \]

Parabolic \[ \kappa_1 \kappa_2 = 0 \]

Includes planar case
Weingarten Operator

\[
W = I^{-1} \cdot \Pi
= A^{-T} \cdot A^{-1} \cdot \Pi
= A^{-T} \cdot A^{-1} \cdot A \cdot \Pi' \cdot A^T
= A^{-T} \cdot \Pi' \cdot A^T
\]

If \( \kappa \) and \( \mathbf{u}' \) are an eigenvalue/vector pair of \( \Pi' \)

Then \( \mathbf{u} = A^{-T} \mathbf{u}' \) is an eigenvector of \( W \) with the eigenvalue \( \kappa \)

The eigenvectors are expressed in the original parameterization
Gaussian curvature

- Measure of intrinsic flatness of the surface
  - Imagine flat-landers computing $\pi$ on the surface

\[ K = \kappa_1 \kappa_2 = \det W = \frac{\det II}{\det I} \]
Mean curvature

- Average curvature of the surface
  - Will be zero for minimal surfaces

\[ H = \frac{\kappa_1 + \kappa_2}{2} = \frac{\text{Tr} (I \cdot I^{\ast})}{2 \det I} \]
Parabolic Lines

• Curves on surface where Gaussian curvature is zero
Contours

• Surface normal perpendicular to view direction

• Generator curve: \[ f(u, v) = (\partial_u \mathbf{S}(u, v) \times \partial_v \mathbf{S}(u, v)).\mathbf{v} = 0 \]
Contours

- Surface normal perpendicular to view direction
- Generator curve: 
  \[ f(u, v) = (\partial_u S(u, v) \times \partial_v S(u, v)) \cdot \mathbf{v} = 0 \]
Geodesic Curves

- Given a curve, $C$, on a surface, $S$
  - $C(t) = S(u(t), v(t))$
- The geodesic curvature is
  - $\kappa^2 = \kappa_g^2 + \kappa_n^2$
  - $\kappa_n = \kappa (\hat{N}_s \cdot \hat{N}_c)$
- Separates curvature into
  - What's necessary to stay on surface
  - What's wiggling in tangent plane
- Geodesics are curves with $\kappa_g = 0$
  - Generalize straight lines
  - Locally shortest path between points
  - On a circle they are great arcs

\[
\ddot{u}_q = (I^{-1})_{qp} \frac{\partial S_k}{\partial u_p} \frac{\partial^2 S_k}{\partial u_i \partial u_j} \dot{u}_i \dot{u}_j
\]

\[
\frac{d^2 C}{dt^2} \cdot \frac{\partial S}{\partial u_i} = 0 \quad \forall i
\]
Geodesic Curves
Geodesic Curves
Geodesic Curves

Note integration errors when passing near poles.
Geodesic Curves

Flat w/ bump

Hyperbolic w/ bump
Lines of Curvature

- A line of curvature on a surface is tangent everywhere to one of the principal curvatures
  - Except at umbilic points where the two principal curvatures are equal

Need to check: lines of curvature geodesic?
Implicit Surfaces

\[ \{ x | f(x) = 0 \} \]

\[ N(x) = \frac{\nabla f}{\| \nabla f \|} \]

\[ K_G = \frac{\nabla f \cdot (\nabla \nabla^T f)^* \cdot \nabla f}{\| \nabla f \|^4} \]

\[ K_M = \frac{\nabla f \cdot (\nabla \nabla^T f) \cdot \nabla f - \| \nabla f \|^2 \text{Tr}(\nabla \nabla^T f)}{2\| \nabla f \|^3} \]

\[ \kappa_1 \|_2 = K_M \pm \sqrt{K_M^2 - K_G} \]

See 2005 paper by Ron Goldman