

## Problem Set 3

*Due Date: March 1, 2018*

A reminder about the course collaboration policy: Your work on problem sets and exams should be your own. You may discuss approaches to problems with other students, but as a general guideline, such discussions may not involve taking notes. You must write up solutions on your own independently, and acknowledge anyone with whom you discussed the problem by writing their names on your problem set. You **may not** use papers or books or other sources (e.g. material from the web) to help obtain your solution.

1. W&S Exercise 5.4
2. W&S Exercise 6.2
3. In this exercise, we'll show how a randomized greedy algorithm can also be used to give a  $\frac{3}{4}$ -approximation algorithm for the maximum satisfiability problem. Assume the variables of the instance are  $x_1, \dots, x_n$ ; in iteration  $i$ , we'll set the  $i$ th variable to true or false. Let  $\text{SAT}_i$  denote the total weight of the clauses satisfied by the settings of the variables at the end of iteration  $i$ , and let  $\text{UNSAT}_i$  be the total weight of the clauses that cannot be satisfied given the settings of the variables at the end of iteration  $i$  (that is, the clauses only contain literals from  $x_1, \dots, x_i$  and given the settings of the variables the clauses are not satisfied). Let  $W$  be the total weight of all clauses. Then observe that  $\text{SAT}_i$  is a lower bound on the weight of clauses satisfied by the algorithm, while  $W - \text{UNSAT}_i$  is an upper bound.

Let  $\text{SAT}_{i,t}$  be  $\text{SAT}_i$  with  $x_i$  set true,  $\text{SAT}_{i,f}$  be  $\text{SAT}_i$  with  $x_i$  set false, and similarly for  $\text{UNSAT}_{i,t}$  and  $\text{UNSAT}_{i,f}$ . Let  $B_i = \frac{1}{2}(\text{SAT}_i + (W - \text{UNSAT}_i))$ ,  $B_{i,t} = \frac{1}{2}(\text{SAT}_{i,t} + (W - \text{UNSAT}_{i,t}))$ , and  $B_{i,f} = \frac{1}{2}(\text{SAT}_{i,f} + (W - \text{UNSAT}_{i,f}))$ . The bound  $B_i$  is midway between our lower and upper bounds on the total weight that the algorithm will satisfy. We'd like to set the next variable so as to increase this bound. In iteration  $i$ , we'll compute  $t_i = B_{i,t} - B_{i-1}$  and  $f_i = B_{i,f} - B_{i-1}$ . If  $t_i \geq 0$  and  $f_i < 0$ , we set  $x_i$  true. If  $t_i \leq 0$  and  $f_i \geq 0$ , we'll set  $x_i$  false. Otherwise (if  $t_i > 0$  and  $f_i > 0$ ) we'll set  $x_i$  true with probability  $t_i/(t_i + f_i)$ . Note the similarity with the "double greedy" algorithm for nonmonotone submodular function maximization that we saw earlier.

- (a) Show that for any  $i$ ,  $t_i + f_i \geq 0$ .
- (b) Given an optimal solution  $x^*$ , let  $\text{OPT}_i$  be the assignment in which  $x_1, \dots, x_i$  are set according to the algorithm and  $x_{i+1}, \dots, x_n$  are set as in  $x^*$ . Let

$w(\text{OPT}_i)$  be the weight of the clauses satisfied by  $\text{OPT}_i$ . Then show that for any  $i$ ,

$$E[w(\text{OPT}_{i-1}) - w(\text{OPT}_i)] \leq \max\left(0, \frac{2t_i f_i}{t_i + f_i}\right).$$

(c) Show that for any  $i$ ,

$$E[w(\text{OPT}_{i-1}) - w(\text{OPT}_i)] \leq E[B_i - B_{i-1}].$$

(d) Show that the algorithm is a  $\frac{3}{4}$ -approximation algorithm for MAX SAT.