Segmentation of Subspace Arrangements
II – GPCA

Allen Y. Yang

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Representation of Subspace Arrangements

1. For a single subspace $V \subset \mathbb{R}^D$, if $\dim(V) = d$ and $r = D - d$:

$$
(u_1^T z = 0) \land (u_2^T z = 0) \land \cdots \land (u_r^T z = 0) \iff \begin{cases} 
  u_1^T z = 0 \\
  \vdots \\
  u_r^T z = 0 
\end{cases}
$$

2. For a subspace arrangement $\mathcal{A} = V_1 \cup V_2 \cup \cdots \cup V_K$,

$$(V_1^\perp)^T z = 0) \lor (V_2^\perp)^T z = 0) \lor \cdots \lor (V_K^\perp)^T z = 0).$$

This constraint can also be written as a system of polynomial constraints:

**Example (Hyperplane Arrangements)**

- A hyperplane is a subspace of $D - 1$ dimension $\Rightarrow r = 1$.
- For a hyperplane arrangement $\mathcal{A} = V_1 \cup V_2 \cup \cdots \cup V_K$:

$$
(u_{1,1}^T z = 0) \lor (u_{2,1}^T z = 0) \lor \cdots \lor (u_{K,1}^T z = 0),
$$

$$
\Rightarrow (u_{1,1}^T z)(u_{2,1}^T z) \cdots (u_{K,1}^T z) = 0.
$$

- $p(z) \doteq (u_{1,1}^T z)(u_{2,1}^T z) \cdots (u_{K,1}^T z)$ is degree-$K$ homogeneous polynomial.
- For any polynomial vanishing on $\mathcal{A}$ (i.e., $\forall z \in \mathcal{A}, p'(z) = 0$), $p'(z) = p(z)g(z)$ for some polynomial $g$.
Example (Point Clusters)

- Point clusters can be treated as *zero-dimensional affine* subspaces.

- Recall the standard procedure: Homogenization.
  For the 1-D case, change the sample coordinates to: \( x = [x, 1]^T \).

- All noise-free samples \( z = [x, y]^T \in V_1 \cup V_2 \) satisfy:
  \[
p(z) = (x - b_1 y)(x - b_2 y) = 0.
\]
Kth Degree Vanishing Polynomials

Example (De Morgan’s Law)

Let $\mathcal{A} = V_1 \cup V_2 \subset \mathbb{R}^3$, $\dim(V_1) = 2$, $\dim V_2 = 1$.

$$(u_{1,1}^T z = 0) \lor \left\{ (u_{2,1}^T z = 0) \land (u_{2,2}^T z = 0) \right\} \Rightarrow \left\{ (u_{1,1}^T z = 0) \lor (u_{2,1}^T z = 0) \right\} \land \left\{ (u_{1,1}^T z = 0) \lor (u_{2,2}^T z = 0) \right\} \Rightarrow \left\{ p_1(z) = (u_{1,1}^T z)(u_{2,1}^T z) = 0 \right\} \land \left\{ p_2(z) = (u_{1,1}^T z)(u_{2,2}^T z) = 0 \right\}

- De Morgan’s law: The constraint for a subspace arrangement can be rewritten as

$$(V_1^\perp z = 0) \lor (V_2^\perp z = 0) \lor \cdots \lor (V_K^\perp z = 0) \Leftrightarrow \left\{ \begin{array}{l} p_1(z) = 0 \\ \vdots \\ p_l(z) = 0 \end{array} \right.,$$

where $\deg(p_i) = K$, and $l \equiv r_1 r_2 \cdots r_K$.

- $p_1, \cdots, p_l$ are all $K$th degree homogeneous polynomials. Define

$$p' = c_1 p_1 + c_2 p_2 + \cdots + c_l p_l,$$

then $\forall z \in \mathcal{A}, p'(z) = 0$.

Any scalar combination also vanishes on $\mathcal{A}$. 
Vanishing Polynomials

**Kth Component of Vanishing Polynomials**
- Denote \( J_K = \text{Span}(p_1, p_2, \cdots, p_l) \). \( J_K \) is a polynomial subspace.
- \( h \doteq \dim(J_K) \), \( h \leq l = r_1 r_2 \cdots r_K \).
- **Completeness:** Given \( K \)th degree homogeneous \( f \), if \( \forall z \in \mathcal{A}, f(z) = 0 \), then \( f \in J_K \).

Vanishing Polynomials
- Define a vanishing polynomial of \( \mathcal{A} \) as \( f(z) = 0 \) for all \( z \in \mathcal{A} \).
- Is it possible that \( \deg(f) > K \)?
- Is it possible that \( \deg(f) < K \)?
- Particularly, given \( p \in J_K \), \( g(z)p(z) = 0 \implies gp \) is a vanishing polynomial.
- All vanishing polynomials form a special polynomial set \( I_\mathcal{A} \), called an *ideal* in algebra.
Properties of Vanishing Polynomials

- \( A \) and \( \mathcal{I}_A \) are completely determined by \( J_K \).
- **[Derksen, 2005]** If \( A = V_1 \cup \ldots \cup V_K \) is in general position, then,

\[
h \doteq \dim(J_K) = \sum_S (-1)^{|S|} \binom{K + D - 1 - c_S}{D - 1 - c_S},
\]

where \( c_S = \sum_{j \in S} c_j \) and the sum is over all \( S \subseteq \{1, \ldots, n\} \) (including the empty set) for which \( c_S < D \).

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Example (\( \dim(J_K) \) for three subspaces in \( \mathbb{R}^3 \))

From Derksen's equation, \( \dim(J_3(A)) \) can only take four possible values:

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Veronese Map

Since $J_K$ completely determines $\mathcal{A}$, we consider estimating $J_K$ from samples.

- $J_K$ is a polynomial subspace, hence only need to get hold of a set of basis vectors.
- The basis vectors for $J_K$ are homogeneous $K$th degree polynomials that are linearly independent.

Example (Basis Vectors of $J_K$)

1. For hyperplane arrangements, $\dim(J_K) = 1$,
   \[
p(z) = (u_{1,1}^T z)(u_{2,1}^T z) \cdots (u_{K,1}^T z).
   \]
   $\Rightarrow J_K = \text{Span}(p)$.

2. Let $\mathcal{A} = V_1 \cup V_2$, then $u_{1,1} = [0, 0, 1]^T$, $u_{2,1} = [1, 0, 0]^T$, $u_{2,2} = [0, 1, 0]^T$, and $\dim(J_2) = 2$.
   \[
p_1 = (u_{1,1}^T z)(u_{2,1}^T z) = x_1 x_3,
   \]
   \[
p_2 = (u_{1,1}^T z)(u_{2,2}^T z) = x_2 x_3.
   \]
   $\Rightarrow J_K = \text{Span}(p_1, p_2)$.

- What is the space of all homogeneous polynomials of degree $K$ containing $J_K$?
  \[
  \mathbb{R}_K[x_1, x_2, \ldots, x_D] \cong \text{Span}(x_1^K, x_2^{K-1}, x_2, \ldots, x_D^K).
  \]
  $\Rightarrow \dim(\mathbb{R}_K[x_1, x_2, \ldots, x_D]) = M_K^{[D]} = \binom{K+D-1}{D-1}$
Definition (Veronese Map)

The Veronese map of order $k$ is the map $\nu_k : \mathbb{R}^D \to \mathbb{R}^{M[D]}_k$ given by

$$\nu_k([x_1, \cdots, x_D]^T) = [x_1^k, x_1^{k-1}x_2, x_1^{k-1}x_3, \cdots, x_D^k]^T,$$

where the list of $x_1^k, x_1^{k-1}x_2, x_1^{k-1}x_3, \cdots, x_D^k$ are all the monomials of degree $k$.

Recovering $J_K$ via the Veronese map

1. Given the number of subspaces $K$ known and $N$ samples $V = \{z_1, \cdots, z_N\}$, construct the data matrix $L_K(V) = [\nu_K(z_1), \nu_K(z_2), \cdots, \nu_K(z_N)] \in \mathbb{R}^{M[D]}_K \times N$.

2. Any $K$th degree vanishing polynomial is expressed by the monomials.

$$p = \begin{bmatrix} c_1, \cdots, c_M[D] \end{bmatrix} \begin{bmatrix} x_1^K \\ x_1^{K-1}x_2 \\ \vdots \\ x_D^K \end{bmatrix}.$$

3. Since $p(z) = 0$ for all $z_1, \cdots, z_N$,

$$\begin{bmatrix} c_1, \cdots, c_M[D] \end{bmatrix} L_K(V) = [p(z_1), p(z_2), \cdots, p(z_N)] = 0_{1 \times N}.$$

4. Hence, the coefficients of a vanishing polynomial as a vector are recovered from $\text{Null}(L_K)$!

$$\dim(\text{Null}(L_K)) = \dim(J_K),$$

and the basis vectors of $\text{Null}(L_K)$ correspond to the basis vectors of $J_K$.
Derivatives of Polynomials

We have learned how to recover vanishing polynomials from the data. Next, how to recover the bases.

- Estimation of the bases for $V_1^\perp, \cdots, V_K^\perp$ depends on the derivatives of the vanishing polynomials.

**Example**

- The null space of $L_2(V)$ is
  
  $c_1 = [0, 0, 1, 0, 0, 0] \Rightarrow p_1 = c_1 \nu_2(x) = x_1 x_3$
  
  $c_2 = [0, 0, 0, 0, 1, 0] \Rightarrow p_2 = c_2 \nu_2(x) = x_2 x_3$

  
  $P(x) = [p_1(x) \ p_2(x)] = [x_1 x_3 \ x_2 x_3]$

- $\nabla_x P = [\nabla_x p_1 \ \nabla_x p_2] = \begin{bmatrix} x_3 & 0 \\ 0 & x_3 \\ x_1 & x_2 \end{bmatrix}$.

- Suppose $z = [1, 1, 0]^T \in V_1$, then $\nabla_x P(z) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$. Suppose $z = [0, 0, 1]^T \in V_2$, then $\nabla_x P(z) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$.

- Hence, $\nabla_x P(z)$ returns a set of basis vectors for $V_1^\perp$ and $V_2^\perp$.

- How to compute derivatives of vanishing polynomials?
  
  Derivatives of monomials are created alongside with the Veronese map:

  $$\nu_k(x) = \begin{bmatrix} x^k \\ x^k_1 x^{k-1}_2 \\ \vdots \\ x^k_D \end{bmatrix} \leftrightarrow \nabla \nu_k(x) = \left[\begin{array}{cccc} x^{k-1} & x^{k-2} & \cdots & 0 \\ 0 & x^{k-1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & x^{k-1}_D \end{array}\right].$$
Given a vanishing polynomial \( p(x) = c_1x^K_1 + \cdots + c_{M[D]}x^K_D, \)

\[
\nabla p(x) = c_1 \nabla x^K_1 + \cdots + c_{M[D]} \nabla x^K_D
\]

\[
= \begin{bmatrix}
x_1^{k-1} & x_2^{k-2} & \cdots & 0 \\
0 & x_1^{k-1} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & x_D^{k-1}
\end{bmatrix}
\begin{bmatrix}
c_1 \\
c_2 \\
\vdots \\
c_{M[D]}
\end{bmatrix}.
\]

Finally, evaluate \( \nabla_x P(z) = \nabla_x [p_1, \cdots, p_h] \) at one point per subspace, we then successfully recover \( V_1^\perp, \cdots, V_K^\perp \)!