CS 294-6 Lecture 12.
Last time.
1. Ambiguities on perspective projection.
   \[ \lambda x' = K \Pi_0 g X = KR_0' R_0 \Pi_0 H_{-1}^H g g_w' g_w X \]

2. Eliminating the ambiguity on \( g_w \): fix the world coord. system on the first camera.
   \[ \begin{cases} \lambda_1 x_1' = K_1 \Pi_0 g_{w1} X_e \\ \lambda_2 x_2' = K_2 \Pi_0 g_{w2} X_e \end{cases} \]
   \[ \Rightarrow (g_{w1} = [I, 0]) \quad \Rightarrow \quad \begin{cases} \lambda_1 x_1' = K_1 \Pi_0 X_e = K_1 \Pi_0 H_{-1}^H X_e \\ \lambda_2 x_2' = K_2 \Pi_0 g_{w2} H_{-1}^H X_e \end{cases} \]

3. Stratification

(1) proj. recon. \( \Rightarrow \) affine recon. \( \Rightarrow \) Euclidean recon.

Let \( H' \) = \[ \begin{bmatrix} K_i' & 0 \\ 0 & 1 \end{bmatrix} \in \mathbb{R}^{4 \times 4}, \text{ then} \]
   \[ \begin{cases} \lambda_1 x_1' = [I, 0] X_e \\ \lambda_2 x_2' = K_2 \Pi_0 g_{w2} H_{-1}^H X_e \end{cases} \]
   \[ = K_2 \Pi_0 g_{w2} \begin{bmatrix} K_i' & 0 \\ 1 & 0 \end{bmatrix} H_{-1}^H X_e \\ = K_2 \Pi_0 g_{w2} H_{-1}^H X_e \\ = K_2 \Pi_0 g_{w2} H_{-1}^H H_{-1}^H X_e \\ = K_2 \Pi_0 g_{w2} H_{-1}^H H_{-1}^H X_e \\ = \Pi_{2p} = K_2 \Pi_0 g_{w2} H_{-1}^H H_{-1}^H X_e \]
   \[ \Rightarrow \begin{cases} \lambda_1 x_1' = \Pi_0 X_p \\ \lambda_2 x_2' = \Pi_{2p} X_p \end{cases} \]

Summary: - projective camera: \( \Pi_{ip} = K_1 \Pi_0 g_{w1} H_{-1}^H H_{-1}^H \]
- affine camera: \( \Pi_{ia} = K_1 \Pi_0 g_{w2} H_{-1}^H \]
- Euclidean camera: \( \Pi_{ie} = K_2 \Pi_0 g_{w2} \)
Projective reconstruction.
Goal: Given point correspondences \(\{(x_1, x_1'), (x_2, x_2')\}\), recover \(\{X_1p, X_2p\}\) and the projection matrix \(\Pi_{1p}\).

Recall we already know how to solve for \(F\).
\[ x_1'^T F x_1 = 0 \iff (x_1' \otimes x_1')^T F^S = 0. \]
Then project \(\tilde{\Omega}_3(F) = 0\).

Also recall such \(F\) is not unique (4-parameter family).
\[ F = T'(KRK^{-1} + V_4 T'V_4). \]

Theorem 6.3 \((\Pi_{1p}, \Pi_{2p})\) and \((\tilde{\Pi}_{1p}, \tilde{\Pi}_{2p})\) yield the same \(F\) matrix iff \(P \in \text{Rg}(\tilde{\Pi}_{1p})\).

\[ \text{proof: first, remember we assume } \tilde{\Pi}_{1p} = [I, 0] \]
\[ \text{denote } \tilde{\Pi}_{2p} = [\tilde{R}', \tilde{T}'] \in \text{Rg}(\Pi_{2p}), \tilde{\Pi}_{2p} = [\tilde{R}', \tilde{T}']. \]
Then, by assumption,
\[ \tilde{R}' \tilde{R} \sim \tilde{R} \tilde{R}' \]
That is, \(A \sim \alpha A\) for some \(\alpha \in \text{R}^\times\).

Notice \(\tilde{T}'\) is in the left null of \(A\),
and \(\tilde{T}'\) is in the left null of \(\alpha A\)
\[ \Rightarrow \tilde{T}' \sim \tilde{T}' \]
\[ \tilde{R}' \tilde{R} \sim (\tilde{R} + \tilde{T}V') \text{ for some } V \in \text{Rg}(\tilde{R}'\tilde{T}'). \]
\[ \Rightarrow [\tilde{R}', \tilde{T}'] \sim [\tilde{R} \tilde{T}'] [I \ 0] \]
\[ \text{full rank} \]

A canonical decomposition.

Since \(F \mapsto (\Pi_{1p}, \Pi_{2p})\) is a one-to-many relation, each map gives a different reconstruction of \(X_{1p}\).

Define (canonical) \(F \mapsto \Pi_{1p} = [I, 0], \Pi_{2p} = [(\tilde{T}')^T F, T] \).

Question: how to get \(T'\)? (unique?) \((\tilde{T}')^T F = 0\)
Reconstruction:
\[
\begin{align*}
\lambda_1 x'_1 &= [I, 0] X_p \\
\lambda_2 x'_2 &= [(T'F, T')] X_p .
\end{align*}
\]
This constraint is linear in \( X_p \).
Solve by SVD. on \( DX_p = 0 \) for a data matrix \( D \).

③ Affine reconstruction
Model: upgrade \( X_p \) to \( X_a \) by
\[
X_a = H_p^{-1} X_p = 
\begin{bmatrix}
I & 0 \\
V_1^T & U_4 \\
0 & 0 \\
U_7 V_4 & 0
\end{bmatrix} \in \mathbb{R}^{4 \times 4} .
\]

* Geometric interpretation of \( [V_1^T, U_4] \).
Notice that when \( [V_1^T, U_4] X_p = 0 \):
\[
\Rightarrow X_a = \begin{pmatrix} X_1 \\ \vdots \\ 0 \\ 0 \end{pmatrix} \text{ affine}
\]
\[
\Rightarrow X_a \text{ is at the infinity}
\]
\[
\Rightarrow \{X_p\} \text{ is called the plane at infinity, } P_\infty.
\]
and denote \( T_{\infty} = [V_1^T, U_4] = [V_1, V_2, V_3, U_4] \in \mathbb{R}^{4} \).
\[
\Rightarrow T_{\infty} X_p = 0 \Rightarrow X_p \in P_\infty
\]

Solution I: Using its vanishing points.
Under perspective projection, \( X_p \) projects onto the image
as vanishing points !
Now, given 3 reconstructed vanishing points \( X'_1, X'_2, X'_3 \),
\[E_v, v_2, v_3, v_4 \] \( X_p = 0 \):
\[
\Rightarrow X_a = 
\begin{bmatrix}
I & 0 \\
V_1^T V_4 & 1
\end{bmatrix} \in \mathbb{R}^{4 \times 4} .
\]

Solution II: Equal modulus constraint.
Sketch: \( \lambda_2 x'_2 = [(T'F - T'V_1^T V_4), T'V_4] X_a \).
Solution III: Can we use mid-point constraints?
\[
\Rightarrow X_a - X'_1 = X_a - X'_2 \]
4. Euclidean Reconstruction (Assume $K_1 = K_2 = K$).

\[
\begin{align*}
\lambda_1 x_1 &= [I, 0] x_a \\
\lambda_2 x_2' &= [K R K^{-1}, K T] x_a
\end{align*}
\]

Goal: Recover the distortion due to the metric $S = K^{-1} T K^{-1}$.

If we denote $K R K^{-1}$ as $\tilde{R}$,

\[
S^{-1} - \tilde{R} S^{-1} \tilde{R}^T = K K^T - \frac{1}{2} K R K^{-1} K K^T R K^{-1} = 0.
\]

This is called a Lyapunov equation.

$L : C^{3x3} \rightarrow C^{3x3}; X \mapsto X - C X C^T$.

We are looking for the kernel of $L$. In fact, symmetric real kernel!

**Theorem 6.9.** Given two matrices $C_i = KR_i K^{-1}$, $i = 1, 2$, where $R_i = e^{\theta_i t}$ with $\|\theta_i\| = 1$, $\theta_i \neq k \pi$, then $SR \ker (L_i) \cap SR \ker (L_2)$ is one-dimensional

iff $U_1$ and $U_2$ are linearly independent.

$\Rightarrow$ for a pair of affine projection matrices $T_1 a, T_2 a$, we can only recover $K$ up to a one-parameter family.

$\Leftrightarrow$ for two pairs of $\{T a\}$, and rotations are along different axes, $K$ can be fully recovered.

**Example 6.10** (pure rotation)

\[
\begin{align*}
\lambda_2 x_1' &= K R K^{-1} x_1, x_1' \text{ (un-calibrated Homography)} \\
\Rightarrow \quad \frac{x_2'}{x_2} K R K^{-1} x_1 &= 0
\end{align*}
\]

Given four points, $(KR, K^{-1}) = \tilde{H}$, can be recovered.

Rotate along another direction $\Rightarrow (KR_2 K^{-1}) = \tilde{H}_2$.

Then, $K$ can be uniquely recovered by $\tilde{H}_1, \tilde{H}_2$. 
4. Calibration with scene knowledge.

1. Suppose the image is a picture of a man-made building.

\[ e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \ e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \ e_3 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \]

and the vanishing points correspondingly are.

\[ v_1 = K e_1, \ v_2 = K e_2, \ v_3 = K e_3. \]

denote

\[ S = K^{-T} K^{-1} \in \mathbb{R}^{3 \times 3}, \]

then

\[ v_i^T S v_j = v_i^T K^{-T} K^{-1} v_j = e_i^T e_j = \delta_{ij}. \]

where, \( \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases} \)

\[ \Rightarrow \text{three independent constraints} \]

\[ \begin{cases} v_1^T S v_2 = 0 \\ v_1^T S v_3 = 0 \\ v_2^T S v_3 = 0 \end{cases} \]

But \( K \) has five parameters:

\[ K = \begin{bmatrix} f & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & g_x & 0 \\ 0 & 0 & 0 & g_y \end{bmatrix} \]

\[ \Rightarrow K \text{ can be recovered up to a 2-parameter family.} \]

To simplify, set \( S_0 = 0 \), and \( S_y = S_y \).

\[ K = \begin{bmatrix} f \Delta & 0 & 0 & 0 \\ 0 & f \Delta & 0 & 0 \\ 0 & 0 & g_x & 0 \\ 0 & 0 & 0 & g_y \end{bmatrix} \]

\[ \text{can be fully recovered.} \]

2. Calibration with a planar pattern (checker board).

Given: \( \checkmark \) multiple images of a checker board.

\( \checkmark \) measurement of the corner points on the board.

\[ \Rightarrow \text{if we set the world coordinate system on the board with } Z = 0, \text{ then we know all points } X_i = \begin{bmatrix} x_i^2 & x_i^2 & x_i^2 & 0 \end{bmatrix} \in \mathbb{R}^4. \]
Hence: \[
\begin{bmatrix}
x_i \\
y_i
\end{bmatrix} = K \begin{bmatrix} x_i' \\
y_i'
\end{bmatrix}
\]
where \(x_i, y_i \in \mathbb{R}^3\) are the first two columns of \(R\).

\[\Rightarrow \hat{x}_i \mathbf{H} [x_i, y_i, 1]^T = 0, \quad \mathbf{H} = K [x_i, y_i, T] \in \mathbb{R}^{3 \times 3}\]

* This is a homography relation between \(X\) in space and its image \(X'\), different from \(\hat{x}_i \mathbf{H} x_i = 0\), whereas both \(x_i\) and \(x_2\) are images.

With more than 4 points, \(K [x_i, y_i, T]\) can be fully recovered, up to a scale factor.

Next, recover \(K, x_i, y_i,\) and \(T\)

let \([c_1, h_2]^T = [K x_i, K y_i]\), then

\[
\begin{align*}
\begin{bmatrix} x_i^T \end{bmatrix} & \mathbf{K}^{-1} \mathbf{K}^{-1} h_2 = 0 \\
\begin{bmatrix} h_1^T \end{bmatrix} & \mathbf{K}^{-1} \mathbf{K}^{-1} h_1 = h_2^T \mathbf{K}^{-1} \mathbf{K}^{-1} h_2
\end{align*}
\]

= Two linear constraints in terms of \(S = \mathbf{K}^{-T} \mathbf{K}^{-1}\).
Since \(S\) has 5 parameters, we need 3 images to fully recover \(S = \mathbf{K}^{-T} \mathbf{K}^{-1} \in \mathbb{R}^{3 \times 3}\).

\(S\) from \(S\) to \(K\)? Cholesky factorization.

\[
S = \begin{bmatrix} \mathbf{S} \\
& 0
\end{bmatrix}
\]

After \(K\) is recovered, we can get hold of \(x_i, y_i,\) and \(T\).