Review of last time

1. Pin-hole camera model

\[ \lambda \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} Sx & S_0 & O_x \\ 0 & S_y & O_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} f & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} \]

unknown depth \quad K \quad \text{intrinsic parameter matrix} \quad T_{i0} \quad \text{projection matrix}

2. Vanishing points and Vanishing lines

1. Let \( x' \) on a line \( L_1 \), \( x^2 \) on \( L_2 \).

\[ \begin{align*}
&\{ X' = X_0 + \mu' V, \mu' \in \mathbb{R}, \text{ then after projection} \\
&\quad X^2 = X_0 + \mu^2 V, \mu^2 \in \mathbb{R}
\end{align*} \]

\[ \begin{align*}
&\lambda X' = K T_{i0} X' \\
&\lambda X^2 = K T_{i0} X^2, \quad \text{let } \mu' \to \infty, \mu^2 \to \infty.
\end{align*} \]

the intersection \( X' \sim K T_{i0} V \).

2. All vanishing points of parallel lines on a plane align on a line in the image plane, called the \textit{vanishing line}.

The vanishing line is the projection of the line at infinity of the plane (horizon).

Suppose \( X_0 \) and \( X_0 \) are two vanishing points for the same plane in space.

\( l_0 = X_0 \times X_0 = X_0 \times X_0 \)
3. Epipolar Constraint: $E = \hat{T} R$, $x^T_2 E x_1 = 0$

Notice that $e_1, e_2, l_2$ are abuse of notation.

$l_1 = \hat{x}_1 e_1$, $l_2 = \hat{x}_2 e_2$, they are co-images.

1. What is $e_1, e_2$: epipoles.

$e_1$ is the image of $O_2$ on $I_1$.
$e_2$ is the image of $O_1$ on $I_2$.

Remember the r.b.m.

$x_{02} = R x_{01} + T$

$\therefore x_{02}(0_1) = R \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + T = T$.

$\therefore \lambda e_2 = KT \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, if we assume calibrated.

$k = I$, then $\lambda e_2 = T$.

Similarly:

$[\begin{bmatrix} T \\ 0 \end{bmatrix}]^{-1} = [\begin{bmatrix} \hat{T} \\ 0 \end{bmatrix}^{-1}$

$\therefore \lambda e_1 = -RT$

$\Rightarrow E e_1 = \hat{T} R e_1 = \hat{T} R (T RT) = 0 \in R^3$

$e_2^T E = T^T \hat{T} R = 0 \in R^3$. 
2. \[ \begin{align*}
& l_1^T x_1 = 0 \Rightarrow l_1 \sim E^T x_2 \\
& x_2^T E x_1 = 0 \\
& x_1^T x_2 = 0 \Rightarrow l_2 \sim E x_1
\end{align*} \] (epipolar lines)

3. Given multiple points on the rigid body \( \{X_1, \ldots, X_N\} \) we have corresponding \( \{X'_1, \ldots, X'_N\} \) on \( I_1 \) and \( \{X_2', \ldots, X_2'\} \) on \( I_2 \).

satisfy
\[ l_1^T x'_1 = 0 \]
\[ l_2^T x'_2 = 0 \]

* However, \( e_1 \) and \( e_2 \) are uniquely determined by \((R, T)\), for all \( l_1 \), \( l_1^T e_1 = 0 \)
\[ l_2 \], \( l_2^T e_2 = 0 \)

So what is \( \text{Span} \{ l_1, \ldots, l_1^N \} \)
\[ S_1 = \text{Span} \{ l_1', \ldots, l_1'^N \} \]
\[ S_2 = \text{Span} \{ l_2', \ldots, l_2'^N \} \]

\( S_1 \perp e_1, \quad S_2 \perp e_2 \)

4. How to solve \( e_1, e_2 \) (ideally)?

* given pairs \((X_1, X_1'), \ldots, (X_N, X_N')\).

Step 1. Recover \( E \) (8-pivot algorithm)

Step 2. Recover \( \{l_1, \ldots, l_1^N\} = S_1 \)
\[ \{l_2, \ldots, l_2^N\} = S_2 \]

Step 3: \( \text{Null}(S_1) = \text{Span} \{ e_1 \} \) (use SVD!!)
\[ \text{Null}(S_2) = \text{Span} \{ e_2 \} \]
4. SVD of $E$ matrix:
A non-zero matrix $E$ is an essential matrix iff $E = U \Sigma V^T$, such that, $U, V \in SO(3)$, and $\Sigma = \text{diag}(0, \sigma, 0)$.

TODAY

1. Eight-point algorithm: solve for $E$ matrix

$$E = TR = \begin{bmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33} \end{bmatrix} \in \mathbb{R}^{3 \times 3}.$$ 

Constraints:

1. given a pair $(x_i, x_j), x_i^TEx_j = 0 \in \mathbb{R}$
2. rank$(E) = 2$ or $\det(E) = 0$

To solve $8$ unknowns, we need $8$ pairs of correspondences in general position in 3-D.

Degenerate cases?

- Points all lie on a planar structure.

- Homography

$\bar{1}$ Rewrite $x_i^TEx_j = 0$ in vector form.

Observe: $x_i^TEx_j = (x_i \otimes x_j)^T E \bar{S}$

where $E \bar{S}$ (stacked) = 

$$\begin{bmatrix} e_{11} \\ e_{21} \\ e_{31} \\ e_{12} \\ e_{22} \\ e_{32} \\ e_{13} \\ e_{23} \\ e_{33} \end{bmatrix} \in \mathbb{R}^9.$$ 

$x_i \otimes x_j$ (Kronecker product) = $[x_i, x_j, \ldots, x_i^T x_j, \ldots, x_i^T x_j] \in \mathbb{R}^{9 \times 9}$

$\therefore (x_i \otimes x_j)^T E \bar{S} = 0$

$\bar{2}$ Solve for the null space of a data matrix.

$$D = \begin{bmatrix} (x_i \otimes x_i')^T \end{bmatrix} \in \mathbb{R}^{N \times 9}, \text{ for } N \text{ pairs of points corres.}$$
\[ \Rightarrow \text{DE}^s = 0 \], and \( E^s \in \mathbb{R}^9 \) is in the 1-D null space of D.
\[ [U, S, V] = \text{svd}(D) \]
\[ E^s \sim V(\text{end}) \), denote its "unstacked" as \( F \).

(3) Projection onto the essential space.
Recall properties of \( E \):
1. \( \text{R} \cdot E = \tilde{R} \cdot \tilde{E} \)
2. \( \sigma_1(E) = \sigma_2(E) = \sigma_3(E) = 0 \).

But \( F \) obtained in (2) is not constrained!!!
That is \( F \& E \), where \( E \) is the space of all essential matrices.

Let \( \text{svd}(F) = U \begin{pmatrix} \sqrt{\lambda_1} & \sqrt{\lambda_2} & \sqrt{\lambda_3} \\ 0 & 0 & 0 \end{pmatrix} V^T \)

Define its projection
\[ E = U \begin{pmatrix} \sqrt{\lambda_1} & \sqrt{\lambda_2} & 0 \\ 0 & 0 & 0 \end{pmatrix} V^T \]

Then \( E \in E \).

**Theorem 5.9.** \( E = \arg \min_{E' \in E} ||F - E'||^2 \).

(4) Recover \( R \) and \( T \).
\[ R = U R^T \begin{pmatrix} \pm \frac{\pi}{2} \\ 0 \end{pmatrix} V^T \]
\[ T = U R^T \begin{pmatrix} \pm \frac{\pi}{2} \end{pmatrix} \Sigma U^T \]

# End of the eight-point algorithm.

**Discussion:**
1. Number of points: up to a scale, \( E \) has 8 unknowns.
   But \( E \) only has 5 DOF: 3 rotation and 2 translation.
   So called "X-point" algorithm.
7-point: consider an extra condition $\det(E) = 0$
we may only use 7 pairs of correspondence.

$$D = \begin{bmatrix} (x_i^1 \otimes x_i^1) \quad \cdots \quad (x_i^9 \otimes x_i^9) \end{bmatrix} \in \mathbb{R}^{7 \times 9}.$$

Let $\{E^5, E^2\}$ be a basis for $\text{Null}(D)$.
Then $\exists$ a unique $d \in \mathbb{R}$, such that:

$$\det(E + dE^2) = 0 \Rightarrow E = E_1 + dE^2.$$

6-point: HW 5-13.

5-point: [Kruppa 1913] But not closed-form.

2) Number of solutions:
In total, there are four possible solutions for $(R, T)$. But only one of them guarantees positive depths of all the 3D points. (HW 5.11)

3) Infinitesimal viewpoint change
\[ T = 0 \iff E = TR = 0. \]
Ideally, 8-point algorithm should return 0.
But due to data noise, $E \neq 0$, $T$ obtained is meaningless.
$\Rightarrow$ Use continuous epipolar constraint.
This is the situation in a video sequence taken by a moving camera.

4) Multiple rigid bodies
We now have $K$ motions: $E_1, \ldots, E_K$ multiple-motion problem
2. Solving for unknown depths $\lambda_i$ after $(R, T)$ are recovered.

$$\lambda_2 x_2 = \lambda_1 R x_1 + \gamma T$$

where $\gamma$ is the scale of the translation. (also unknown)

If we assume the scale of 3-D is 1.

$$\lambda_1 x_2 R x_1 + \lambda_2 T = 0$$

$$\Rightarrow \begin{bmatrix} x_2 R x_1 & \lambda_2 T \\ \lambda_1 & 1 \end{bmatrix} = 0$$

Condition on that

$$\begin{bmatrix} x_2 R x_1 & \lambda_2 T \end{bmatrix}$$

is a rank-1 matrix.


Goal: Given noisy image pairs $(\tilde{x}_1, \tilde{x}_2)$, solve for the optimal $E^*$ to approximate the epipolar relation and recover the noiseless positions $(x_1, x_2)$

1. Noise model

$$\tilde{x}_1 = x_1^t + w_1^t; \quad \tilde{x}_2 = x_2^t + w_2^t, \quad j = 1, 2, \ldots, N,$$

$$w_1^t = \begin{bmatrix} \omega_1^t \\ \omega_1^t \end{bmatrix}, \quad w_2^t = \begin{bmatrix} \omega_2^t \\ \omega_2^t \end{bmatrix}$$

are localization error.

$$. x_2^t R x_1 = 0, \text{ but } x_2^t T R x_1 \neq 0.$$

2. Objective function

$$\phi(x_1, R, T, \lambda_1) = \sum_{j=1}^{N} \left[ \left\| \tilde{x}_1 - x_1^t \right\|^2 + \left\| \tilde{x}_2 - (R x_1^t + x_1^t) \right\|^2 \right]$$

$$(x_1^t, R^t, T^t, \lambda_1^t) = \arg\min \phi(x_1, R, T, \lambda_1)$$

3. Optimization techniques: