1. Bundle Adjustment

Given the noise model:
\[
\begin{align*}
\tilde{x}_1^j &= x_1^j + w_1^j, \quad j = 1, \ldots, N. \\
\tilde{x}_2^j &= x_2^j + w_2^j
\end{align*}
\]

The objective function is
\[
\Phi(x_1, R, T, x_2) = \sum_{j=1}^{N} \left[ \| \tilde{x}_1^j - x_1^j \|_2^2 + \| \tilde{x}_2^j - T (R x_1^j + T) \|_2^2 \right]
\]

Conditioning that \( x_1^T \Theta [8] = 1, \quad R \in SO(3), \quad \| T \| = 1 \)

Recall: what would happen when \( T \to 0 \).

This optimization is highly nonlinear and expensive.

Solution:

Iteration between optimizing \( (R, T) \) and \( (x_1, x_2) \)

Step 1: given \( (x_1^j, x_2^j) \) for \( j = 1 \), solve for \( R, T \) using 8-point.

Step 2: Optimize \( (R, T) \) by
\[
\Phi_i(R, T) = \sum_{j=1}^{N} \left( \frac{x_3^j \cap R x_1^j}{\| \tilde{x}_3^j \|_2} + \frac{(x_2^j \cap R x_1^j)^2}{\| \tilde{x}_2^j \|_2^2} \right)
\]

Geometric interpretation? To minimize the projected distances.

Step 3: Using \( (R, T) \) updated, optimize
\[
\Phi_c(x) = \sum_{j=1}^{N} \left[ \| x_1^j - \tilde{x}_1^j \|_2^2 + \| x_2^j - \tilde{x}_2^j \|_2^2 \right]
\]

Step 4: if updates for \( R, T, x_1 \) are small, stop.
otherwise, go to Step 2.

Further Reading: Section 5.4.

Implementations for the optimization:
Choice 1: Gauss-Newton Method
Choice 2: Levenberg-Marguardt Method

Brief overview: let the cost function
\[ \Phi(\vec{x}) = \frac{1}{2} \vec{Y}(\vec{x})^T \vec{Y}(\vec{x}) \] (sum of squares)

\[ \nabla \Phi(\vec{x}) = J^T(\vec{x}) \vec{Y}(\vec{x}) \]

\[ H_{\Phi}(\vec{x}) = J^T(\vec{x}) J(\vec{x}) + \sum_{i=1}^{m} R_i(\vec{x}) H_{R_i}(\vec{x}) \]

- 1st-order approximation

Then apply Newton's method, starting with an initial \( \vec{x}_0 \)

\[ \vec{x}_{k+1} = \vec{x}_k + \vec{s}_k \]

where

\[ \vec{s}_k \] is computed by

\[ (J^T(\vec{x}_k) J(\vec{x}_k)) \vec{s}_k = -J^T(\vec{x}_k) \vec{Y}(\vec{x}) \]

Recall:

\[ f(\vec{x} + \vec{s}) \approx f(\vec{x}) + \nabla f(\vec{x})^T \vec{s} + \frac{1}{2} \vec{s}^T H_{\Phi}(\vec{x}) \vec{s} \]

Set \( \frac{\partial f}{\partial \vec{s}} = 0 \) \( \Rightarrow H_{\Phi}(\vec{x}) \vec{s} = -\nabla f(\vec{x}) \]

Homography for planer scenes.

1. for all points \( X \in P \),
   \[ N^T X = d \iff \frac{1}{d} N^T X = 1 \]
   where \( N \) is the unit normal vector of \( P \) with respect to \( O_w \).

Given another vantage point \( O_w' \),

\[ X_2 = RX_1 + T = RX_1 + T \frac{1}{d} N^T X_1 = (R + \frac{1}{d} TN^T) X_1 \]

Call \( H = R + \frac{1}{d} TN^T \in \mathbb{R}^{3 \times 3} \) the homography matrix.
The linear transformation between \( O \) in \( O_2 \) is
\[
X_2 = HX_1 \quad \Rightarrow \quad \tilde{X}_2 = H\tilde{X}_1, \quad (equality \ up \ to \ a \ scalar)
\quad \Rightarrow \quad \tilde{X}_2 HX_1 = 0 \quad \in \mathbb{R}^3
\]

2. Relations between homography and essential matrix. Recall: To recover a correct \( E \) matrix, 3-D points have to be in general position.
When all points lie on a plane, we consider the degenerate situation:

\[\text{Theorem 5.21.} \quad \text{For } E = \hat{T}R \text{ and } H = R + Tu^T \text{ for } T, u \in \mathbb{R}^3 \text{ with } \|T\| = 1,
\]

1. \( E = \hat{T}H \)
2. \( H^T E + E^T H = 0 \)
3. \( H = \hat{T}E + Tu^T \) for some \( v \in \mathbb{R}^3 \).

\[\text{Proof.} \quad 1. \quad \hat{T}H = \hat{T}R + \hat{T}Tu^T = E.
2. \quad H^T E = (R + Tu^T)^T \hat{T}R = R^T \hat{T}R
\]
\[\quad = -(R^T \hat{T}R)^T
\quad = -E^T H.
3. \quad H = \hat{T}E + Tu^T \text{ for some } v.
\Rightarrow \quad (H - \hat{T}E) = Tu^T \text{ for some } v.
\Rightarrow \quad H - \hat{T}E = \begin{bmatrix} v_1^T & v_2^T & v_3^T \end{bmatrix}
\Rightarrow \quad \hat{T}(H - \hat{T}E) = 0
\Rightarrow \quad \hat{T}H = \hat{T}\hat{T}E
\]
\[\text{But } \hat{T}H = \hat{T}R \neq \hat{T}\hat{T}E \quad \Rightarrow \quad \hat{T}H = \hat{T}\hat{T}E \quad \Rightarrow \quad \text{HW 5.3}
\]

Note: \( \hat{T}\hat{T} \) is called an orthogonal projection.
\[
\hat{T}\hat{T}v = \hat{T}(-\hat{T}0) = \hat{T}w = v_1, \quad \|v_1\| = \|v_1\| = 0 \quad \Rightarrow \quad \|T\| = 1
\]

\[
\hat{T}\hat{T} = I - TT^T \quad \text{(Exer 5.3.1)}
\]
Corollary: Given a pair of points \((x_1, x_2)\) and homography 
\[ x_2^* \sim H x_1, \text{ for } \forall u \in \mathbb{R}^3, \]
\[ x_2^* \hat{u} H x_1 = 0, \]
Therefore, a homography \(H\) corresponds to a 3-parameter family of essential matrices \(E = \hat{u} H \in \mathbb{R}^{3 \times 3}\).

Proof: Let \(w = \hat{u} x_2\), \(w \perp x_2 \sim H x_1\)
\[ \therefore w^T H x_1 = 0 \Rightarrow x_2^T \hat{u} H x_1 = 0. \]

Hence, eight-point algorithm does not apply to points from a planar scene.

(3) Estimation of \(H\).
\[ \lambda_2 x_2^* = H \lambda_1 x_1 \]
\[ \iff \hat{x}_2^* H x_1 = 0 \]
\[ \iff (x_1 \otimes \hat{x}_2)^T H^9 = 0 \]
Where \(x_1 \otimes \hat{x}_2 \in \mathbb{R}^{9 \times 3}\), and \(H^9 \in \mathbb{R}^{9}\)

To solve \(H\) up to a scale, we also need 8 linear constraints.
\[ \therefore \text{rank} (\hat{x}_2) = 2 \]
\[ \therefore \text{rank} (x_1 \otimes \hat{x}_2) = 2 \]
\[ \Rightarrow \text{one pair of correspondence provides 2 constraints} \]
Therefore, we need at least FOUR points in space to solve for \(H\).

* four-point algorithm.
\[ D = \begin{bmatrix} (x_1 \otimes \hat{x}_2)^T \\ \vdots \\ (x_4 \otimes \hat{x}_2)^T \end{bmatrix} \in \mathbb{R}^{12 \times 9} \]
\[ \text{rank}(D) = 8 \]
\[ [U, S, V] = \text{svd}(D), \text{ and } H^9 = V(\text{end}). \]
Theorem 5.18: for $H_L = \lambda (R + \frac{1}{\alpha}TN^T)$, $|\lambda| = \sigma_2(H_L)$.

Proof: Let $u = \frac{1}{\alpha}RT^T \in \mathbb{R}^3$.

$$H_L^TH_L = \lambda^2 \left( I + uN^T + Nu^T + uN^T uN^T \right)$$

Let $v = u \times N = uN$, then $\nabla u \cdot v \nabla N$

$$\Rightarrow H_L^TH_L v = \lambda^2 v$$

$\lambda^2$ is an eigenvalue of $H_L^TH_L$.

$$\Rightarrow |\lambda|$$ is a singular value of $H_L$.

Next, we need to prove $|\lambda|$ is the second S.V.

Consider $Q = uN^T + Nu^T + uN^T uN^T$

$$= (uN + uN^T) \left( uN + uN^T \right)^T - uN^T uN^T$$

(denoted) = $(w + v)(w + v)^T - WW^T$

$\Rightarrow \sigma_2(Q) = 0$

(i) if $v \neq w$, let $W = aU$ for $a \in \mathbb{R}$

$Q = (a + 1)^2 vv^T - a^2 WW^T = (2a + 1)U^T$

let a basis in $\mathbb{R}^3$ be $\{v, v_2, v_3\}$,

$$Qv = 0, Qv_2 = 0$$

(ii) if $v = w$, $\exists \theta$, $\theta < 0$.

then $Qv = 0$. $\forall \theta \in \mathbb{R}^3$

denote $v' = w + v$, then

$$Q = v'v'^T - WW^T$$

is a symmetric matrix.

It is easy to see that if $v', w$, then $\exists \theta \in \mathbb{R}^3$

$\text{u}^TQu > 0$

and $\exists \theta \in \mathbb{R}^3$ $\text{u}^TQu < 0$.

$\therefore Q$ has one positive eigenvalue and one negative eigenvalue.

Summary: in both cases, $0$ is $\sigma_2(Q)$

$$\Rightarrow |\lambda| = \sigma_2(H_L)$$
Hence, we set $H = H_u / \Sigma_2(H_u) \Rightarrow \Sigma_2(H) = 1$.

This recovers $H$ up to a sign difference

$$H = \pm (R + \frac{1}{\alpha} T N^T).$$

To recover the sign, use the positive weights constraint

$$\lambda_2 x_2 = H_0 \lambda_1 x_1$$

$$\Rightarrow x^2 H x_1 > 0 \text{ for all pairs } (x^2, x^1)$$

\[5\] Decomposing $H$

Theorem 5.19. Given $H = R + \frac{1}{\alpha} T N^T$, there are at most two possible solutions for a decomposition, from 4 solutions of the following form: (Sketch)

- (i) significance of $\Sigma_2(H) = 1$
  
  $$\exists \text{ a vector } a \in \mathbb{R}^3$$

  $$\|Ha\|^2 = \|a\|^2$$

  It is not surprising that for $a \perp N$, $Ha = Ra$

Hence,

$$\text{SVD } (H^T H) = V \Sigma V^T,$$

$$\Sigma = \text{diag } \{ \sigma_1^2, \sigma_2^2, \sigma_3^2 \}$$

$$H^T H v_1 = \sigma_1^2 v_1, \quad H^T H v_2 = \sigma_2^2 v_2, \quad H^T H v_3 = \sigma_3^2 v_3.$$ 

$$\therefore v_2 = V(:, 2) \perp N$$

Construct

$$u_1 = \frac{\sqrt{1 - \sigma_2^2} v_1 + \sqrt{\sigma_1^2 - 1} v_3}{\sqrt{\sigma_1^2 - \sigma_2^2}}$$

$$u_2 = \frac{\sqrt{1 - \sigma_3^2} v_1 - \sqrt{\sigma_1^2 - 1} v_3}{\sqrt{\sigma_1^2 - \sigma_3^2}}$$

We can check $\|Hu_1\|^2 = \|u_1\|^2$

further, $H$ preserves any vector inside

$$S_1 = \text{span } \{ v_2, u_1 \} \quad \text{and } \quad S_2 = \text{span } \{ v_2, u_2 \}$$

$$\therefore N \perp S_1, \quad N \perp S_2$$

$$\therefore N = \tilde{v}_2 u_1 \text{ or } \tilde{v}_2 u_2.$$ 

\[ii\] Define

$$u_1 = [v_2, u_1, \tilde{v}_2 u_1]; w_1 = [Hv_2, Hu_1, \tilde{v}_2 u_1]$$

$$u_2 = [v_2, u_2, \tilde{v}_2 u_2]; w_2 = [Hv_2, Hu_2, \tilde{v}_2 u_2]$$
Then $RU_1 = W_1$, $RU_2 = W_2$
$\Rightarrow R = U_1 U_1^T$ or $U_2 U_2^T$.

Hence: 4 Solutions:
$R_1 = U_1 U_1^T = R_2$
$N_1 = U_2 U_1$, $\frac{1}{\alpha} T_1 = (H - R_1) N_1$
$N_2 = -N_1$, $\frac{1}{\alpha} T_2 = -\frac{1}{\alpha} T_2$.

$R_3 = U_2 U_2^T = R_4$
$N_3 = U_2 U_2$, $\frac{1}{\alpha} T_3 = (H - R_3) N_3$
$N_4 = -N_3$, $\frac{1}{\alpha} T_4 = -\frac{1}{\alpha} T_3$.

Claim: only two of normals $N_i$ have positive depth.
# end of the four-point algorithm.

Homework #3: 5.3, 5.6, 5.11, 5.13, 5.19

Special Motions
1. Pure rotation, $T=0$, $E = \hat{T} R=0$, but $H=R$
$\Rightarrow X_2 = RX_1 \Leftrightarrow \hat{X_2} RX_1 = 0$
meaning without translation, depth information is completely lost, the 3D scene can be interpreted to be planar.

2. Planar motion: the camera always moves on a plane: (Exercise 5.6)