61A Lecture 21

Monday, October 15

Tree Recursion

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A Tree-Recursive Process

The computational process of fib evolves into a tree structure

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```
fib(6)
```

A Tree-Recursive Process

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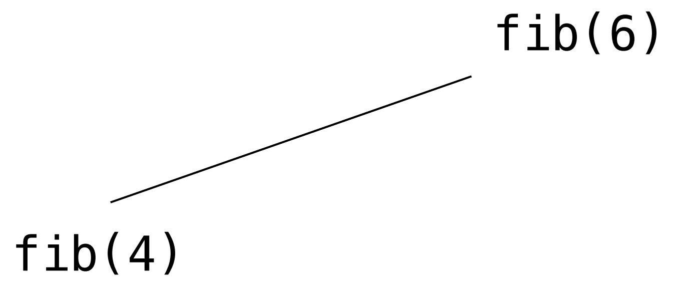

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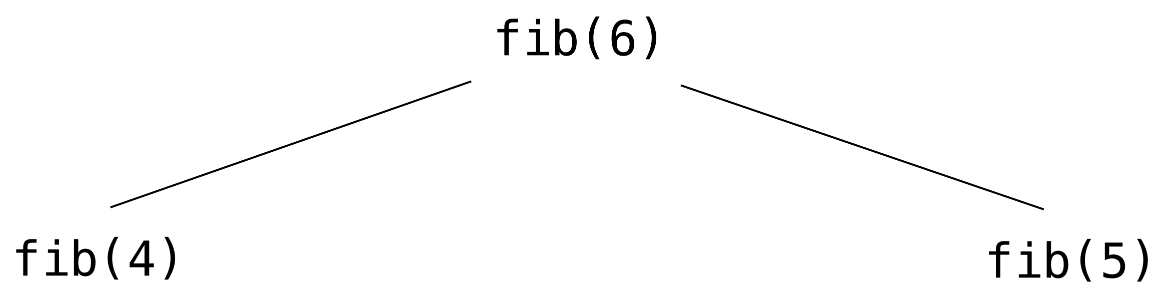

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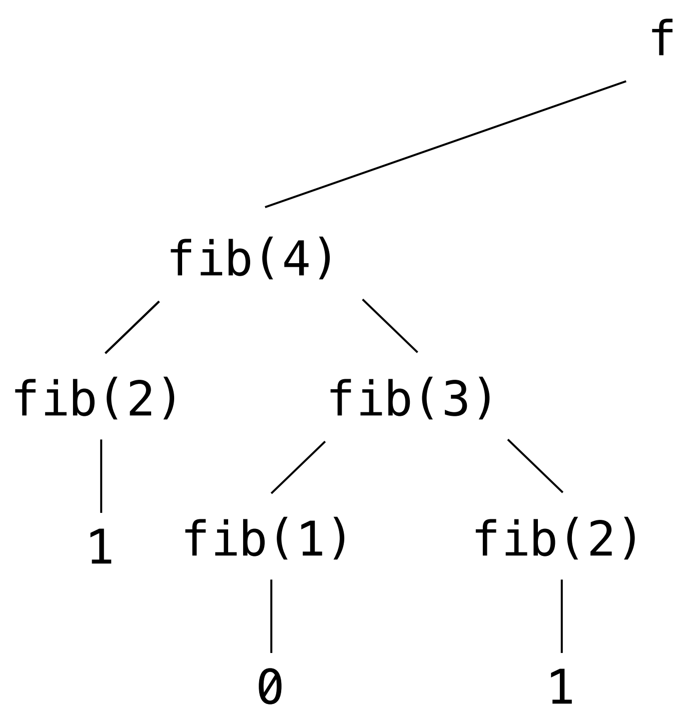

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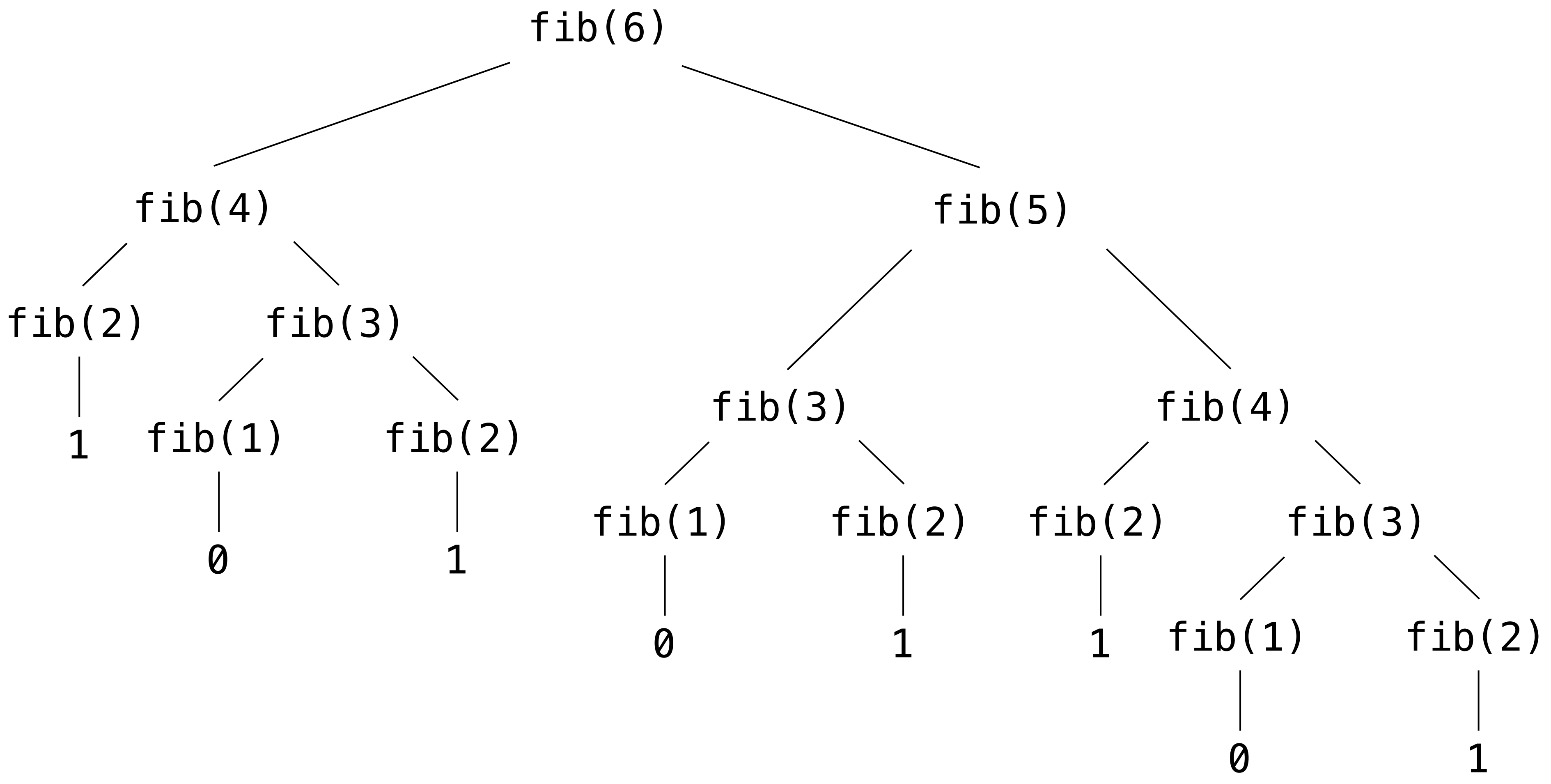

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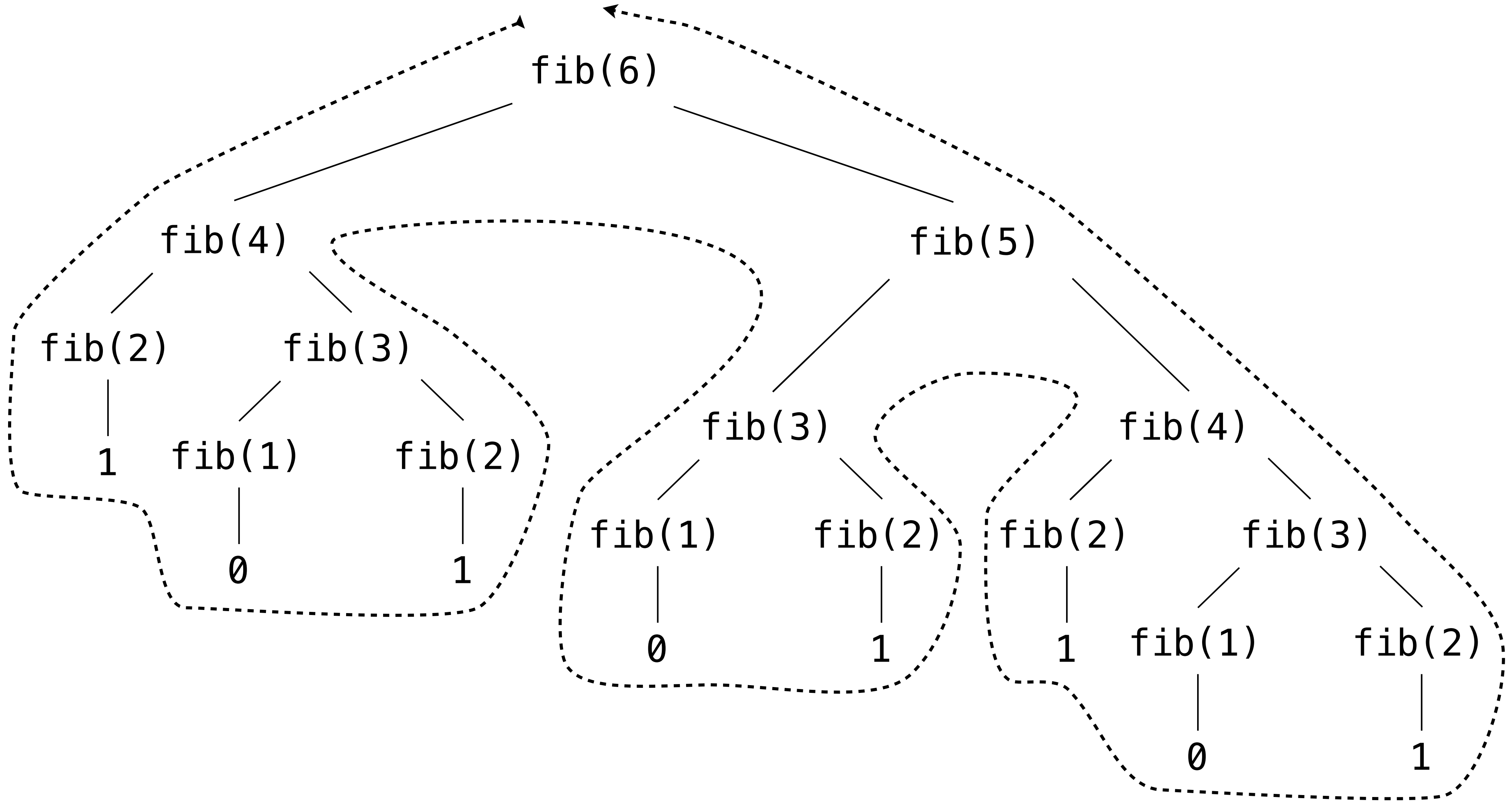

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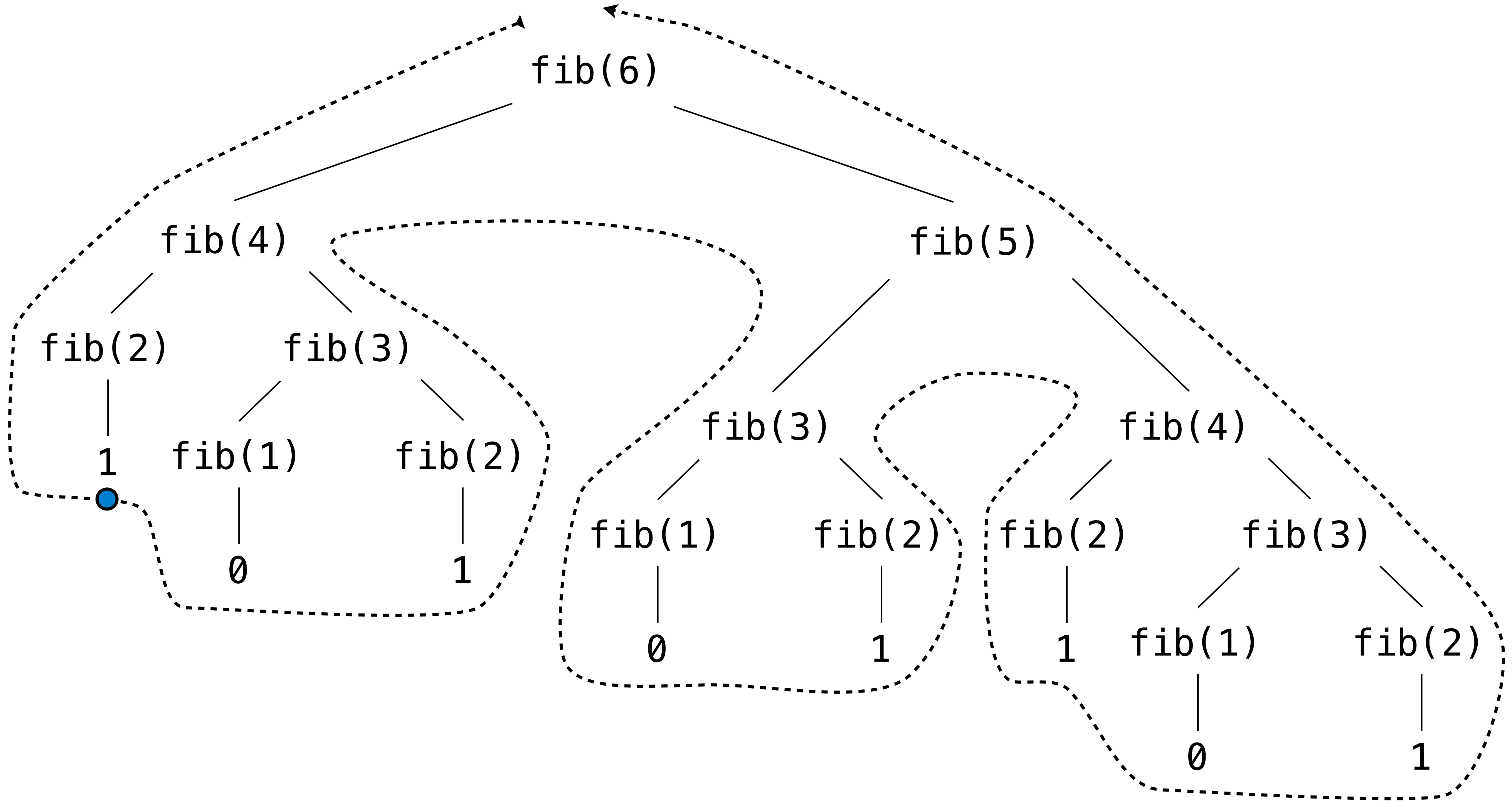

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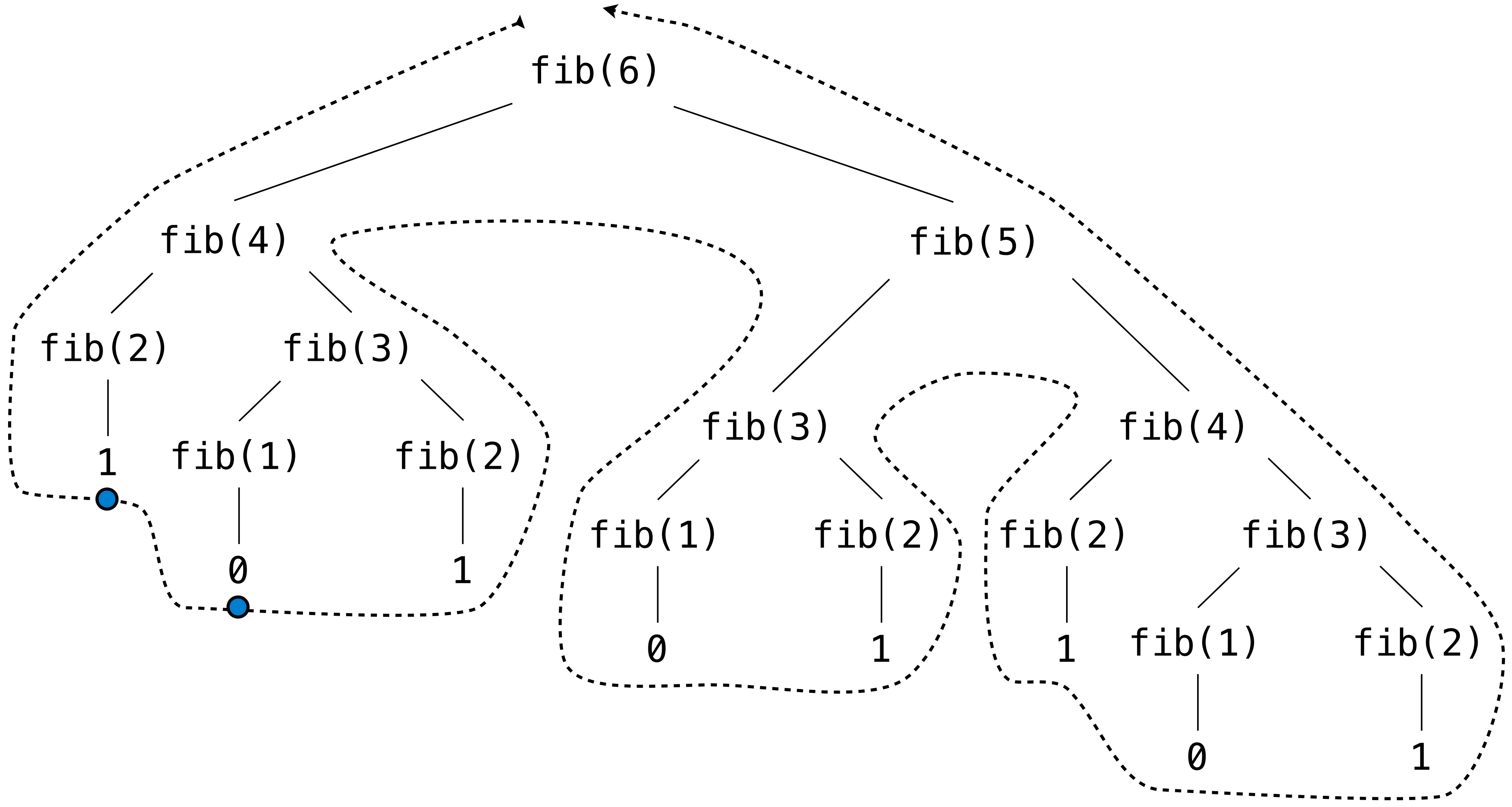

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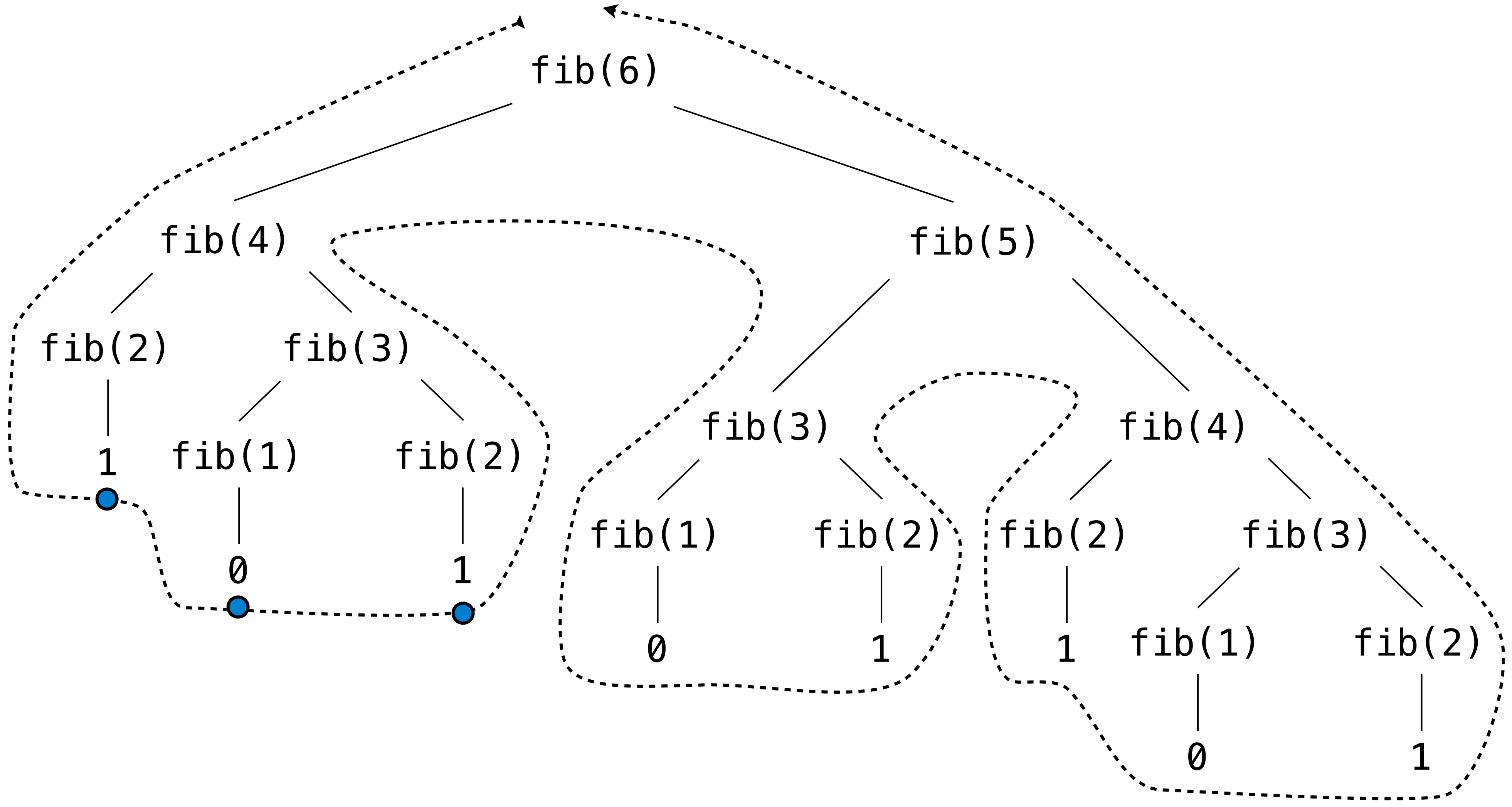

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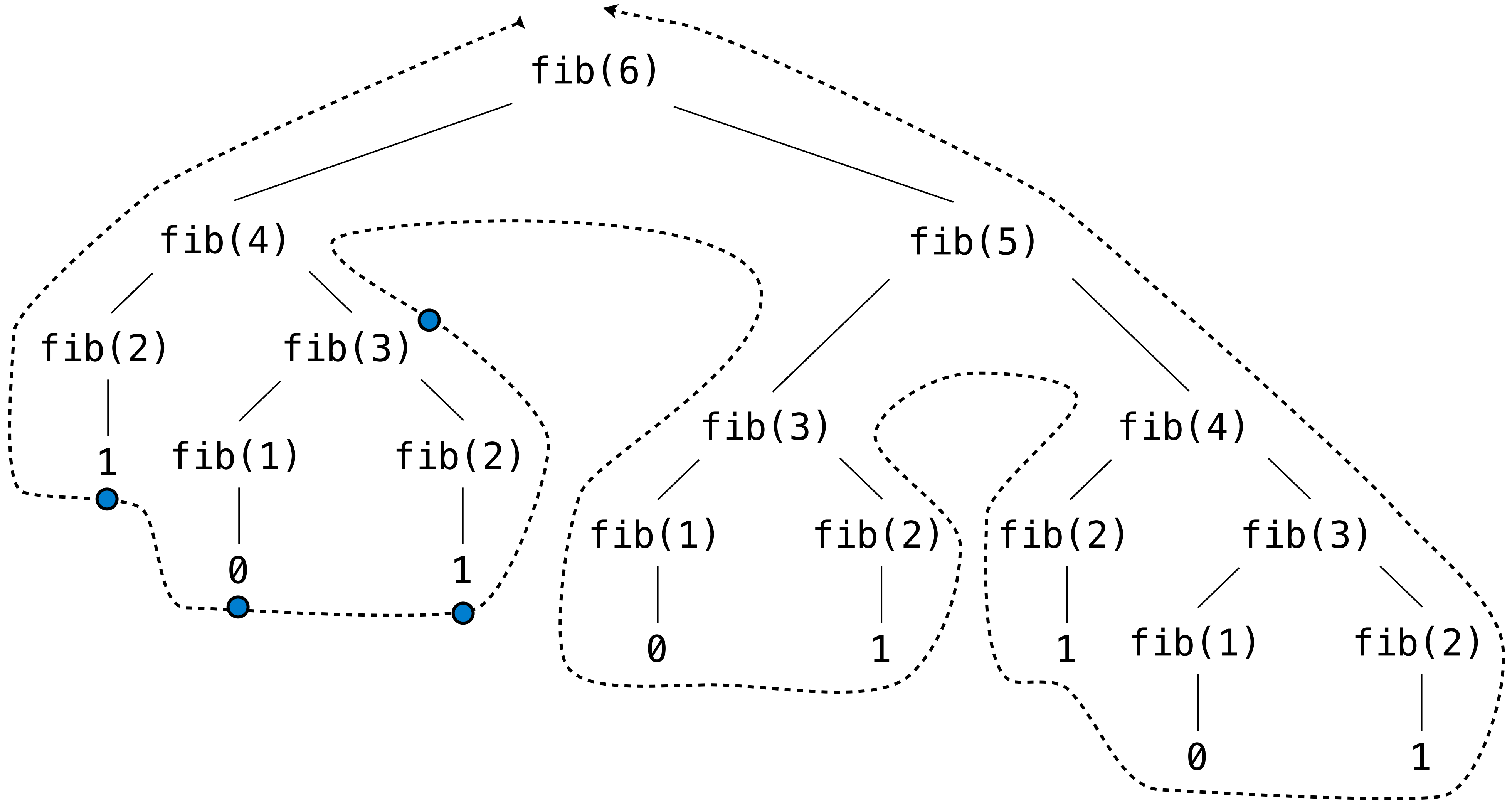

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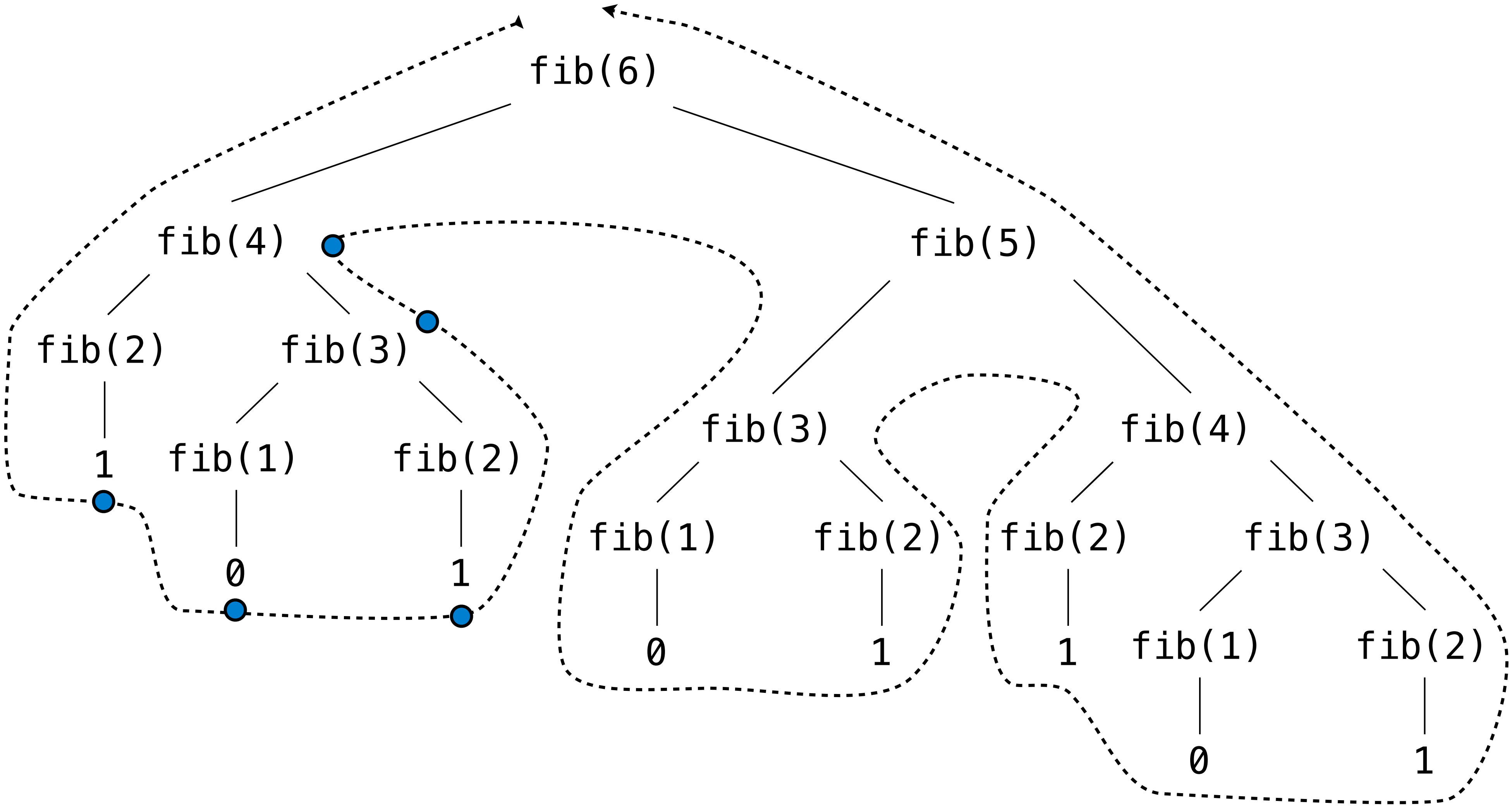

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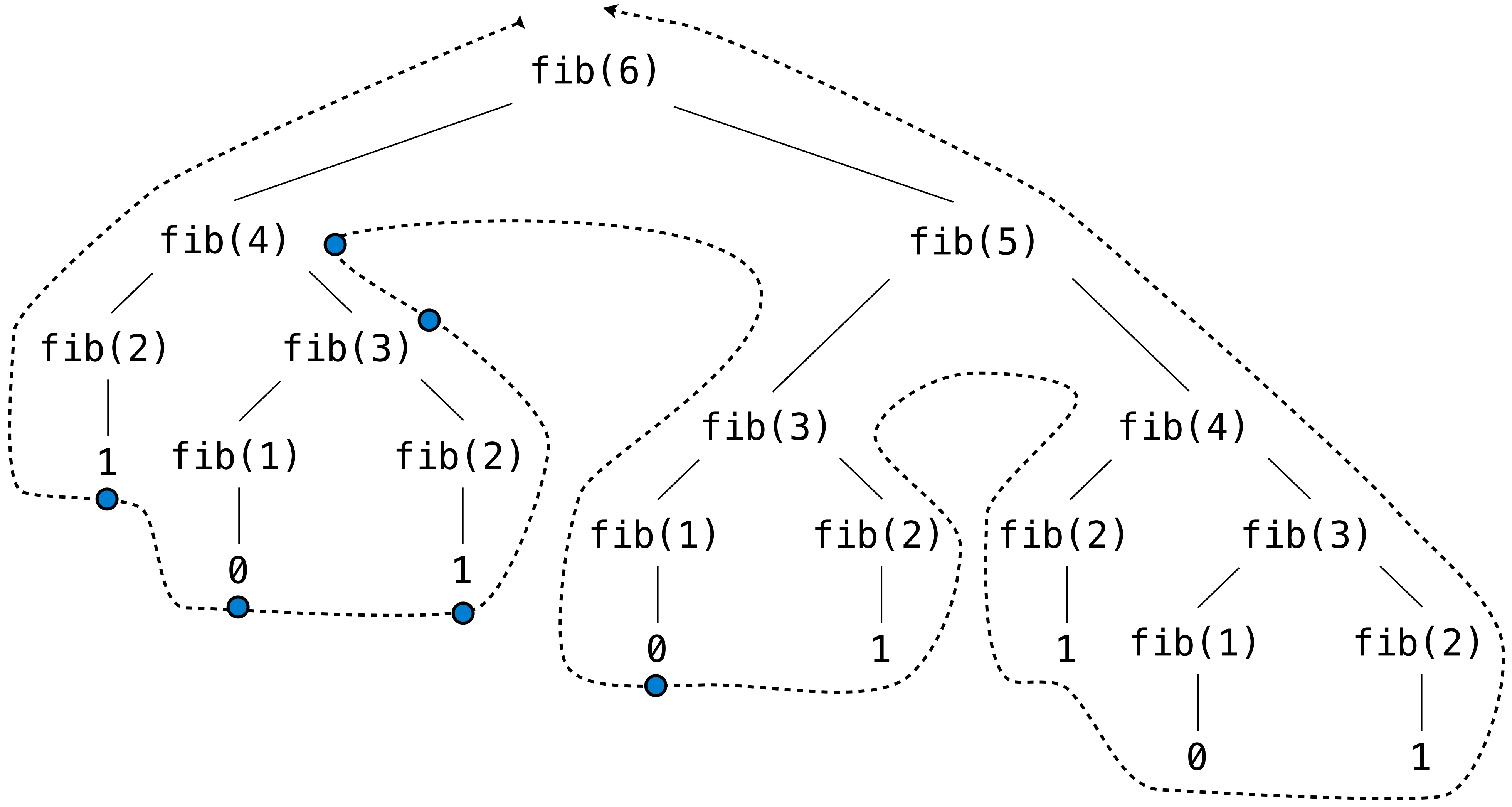

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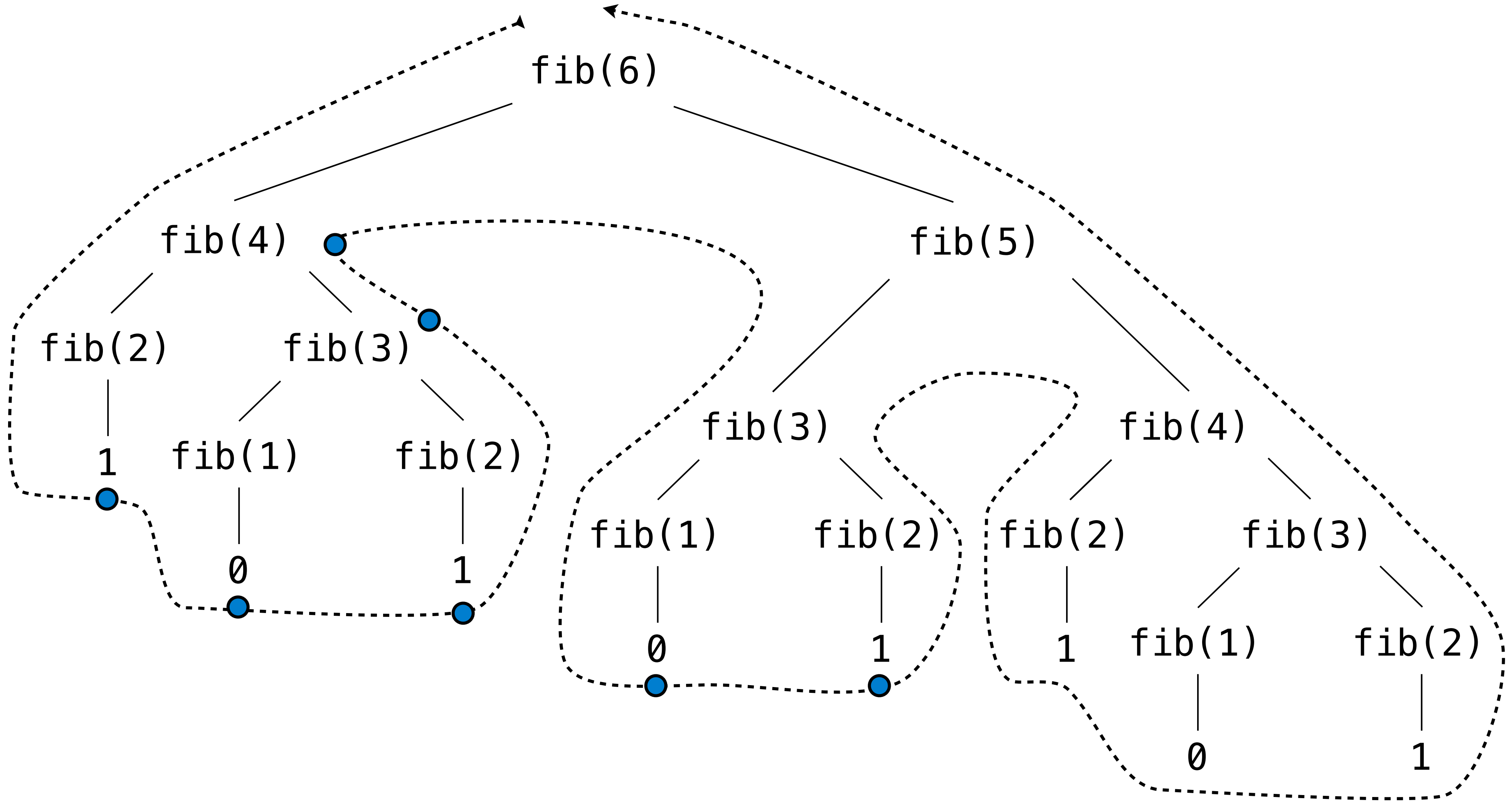

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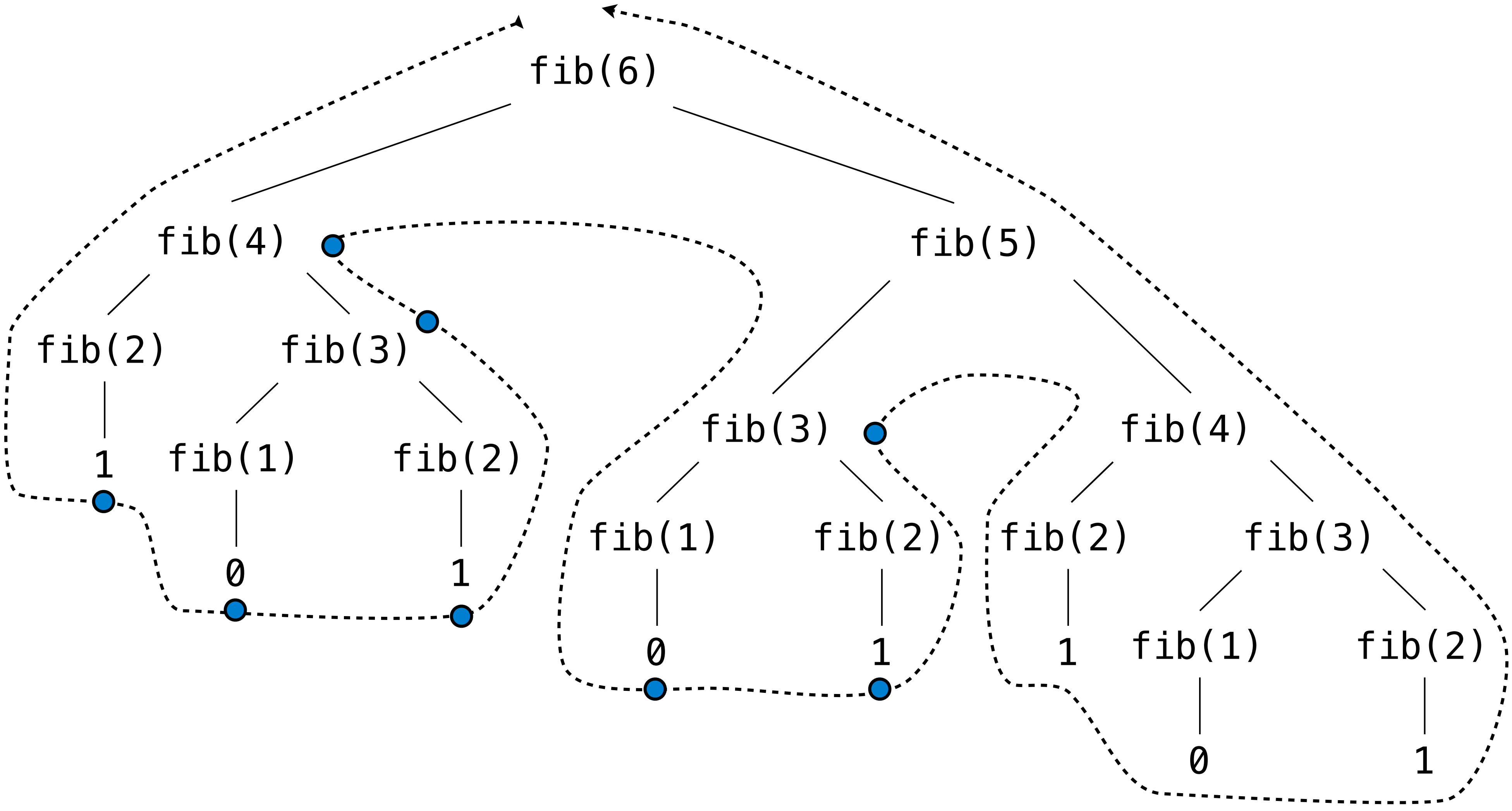

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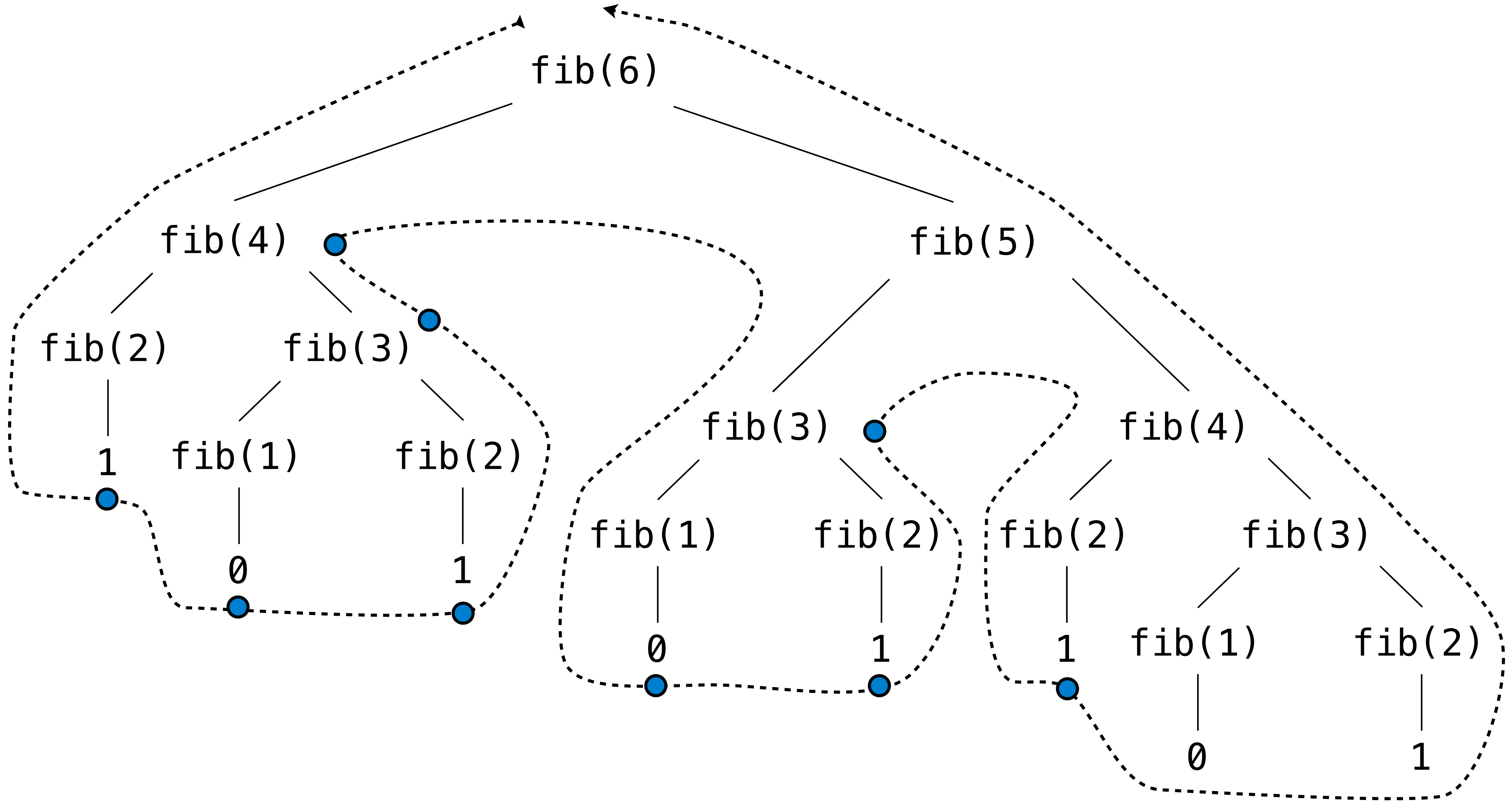

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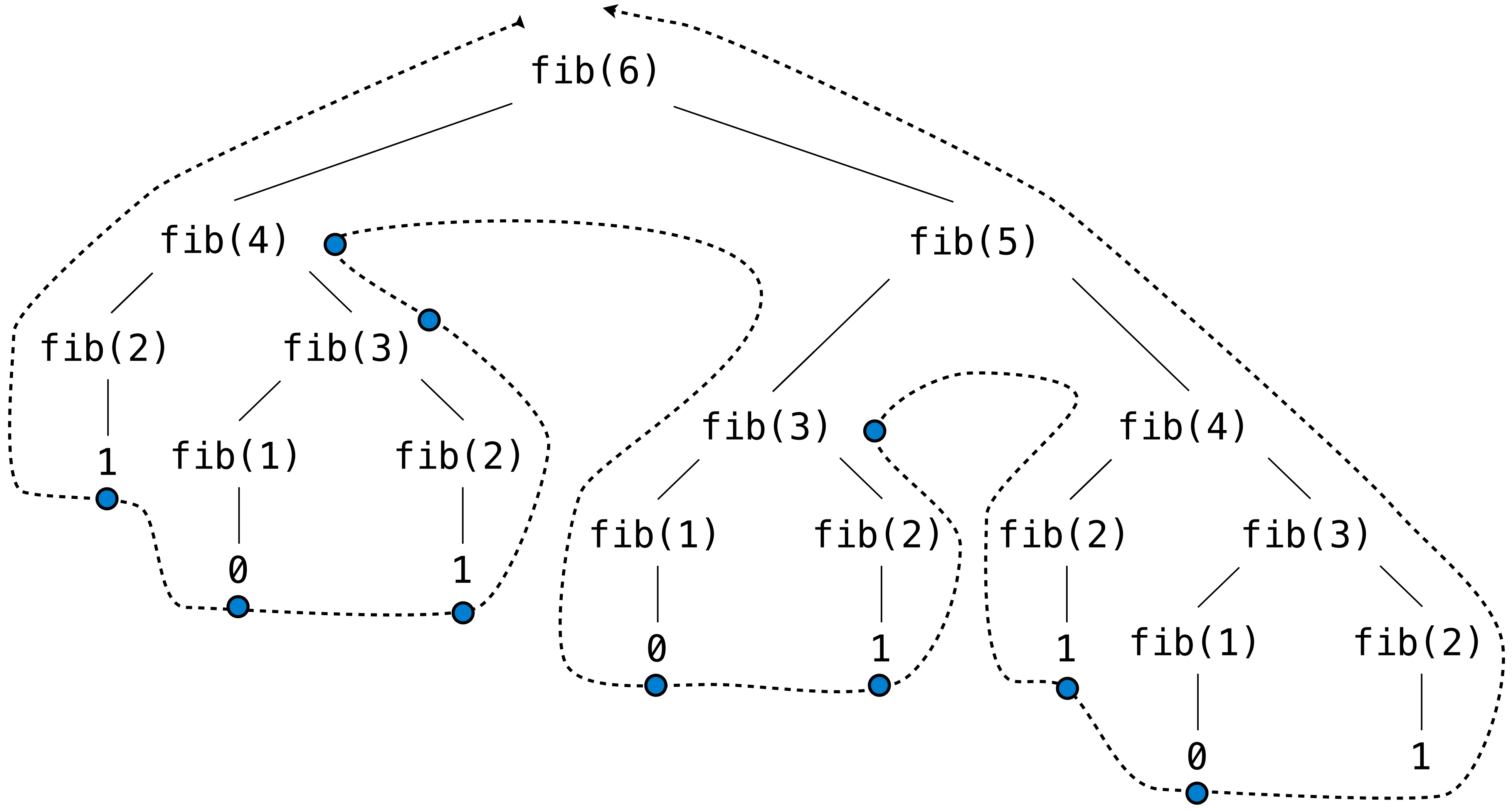

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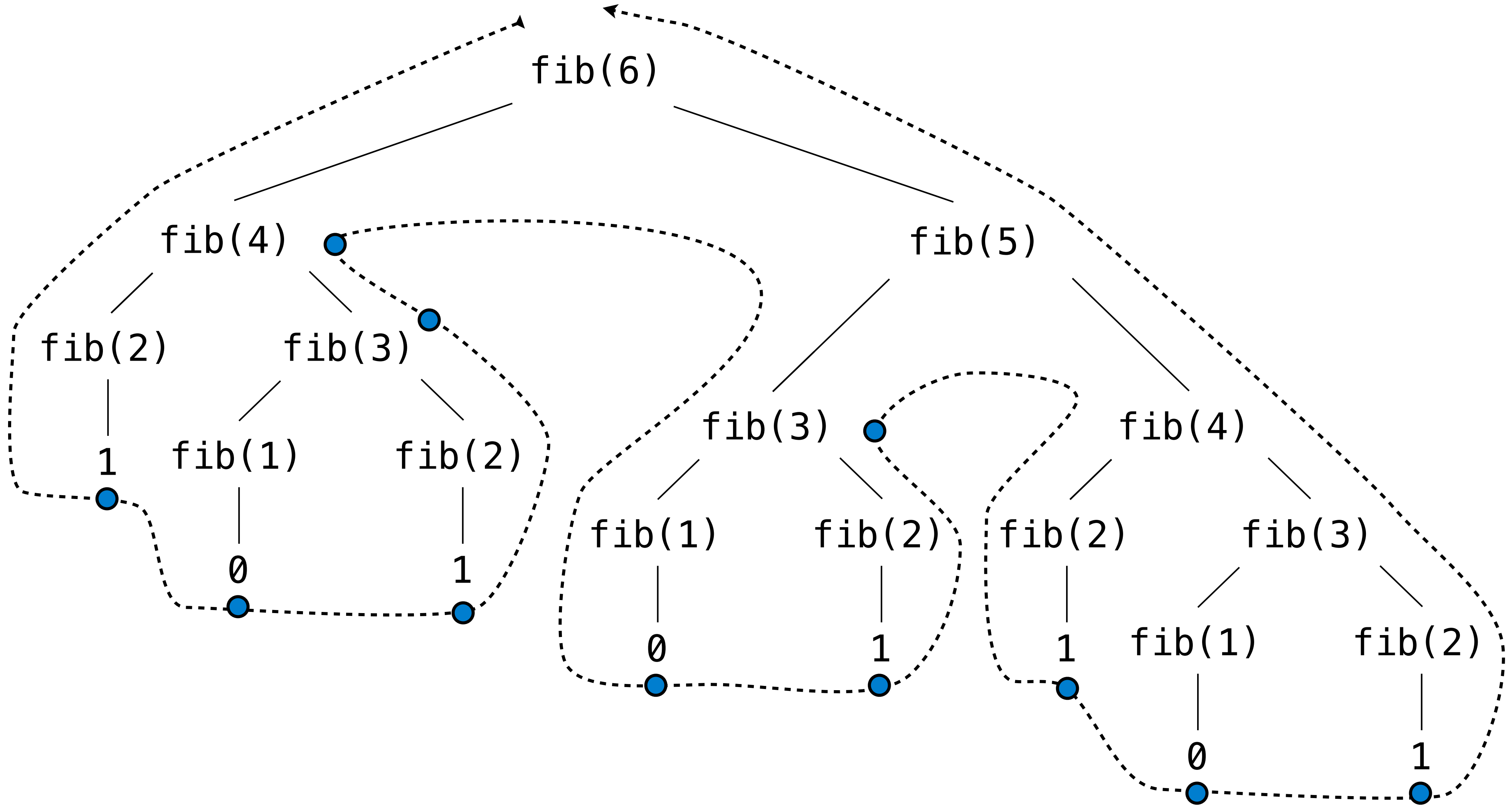

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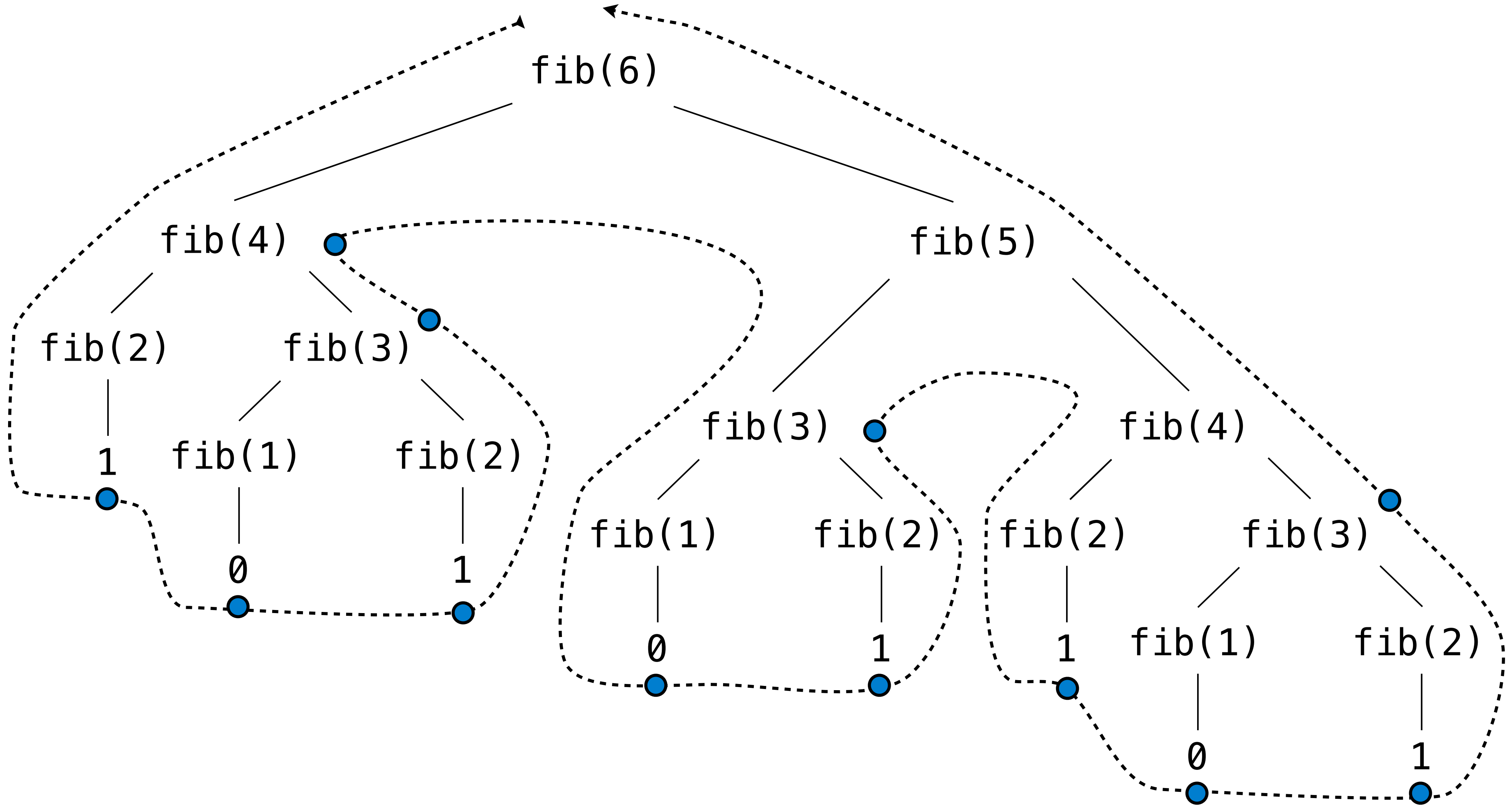

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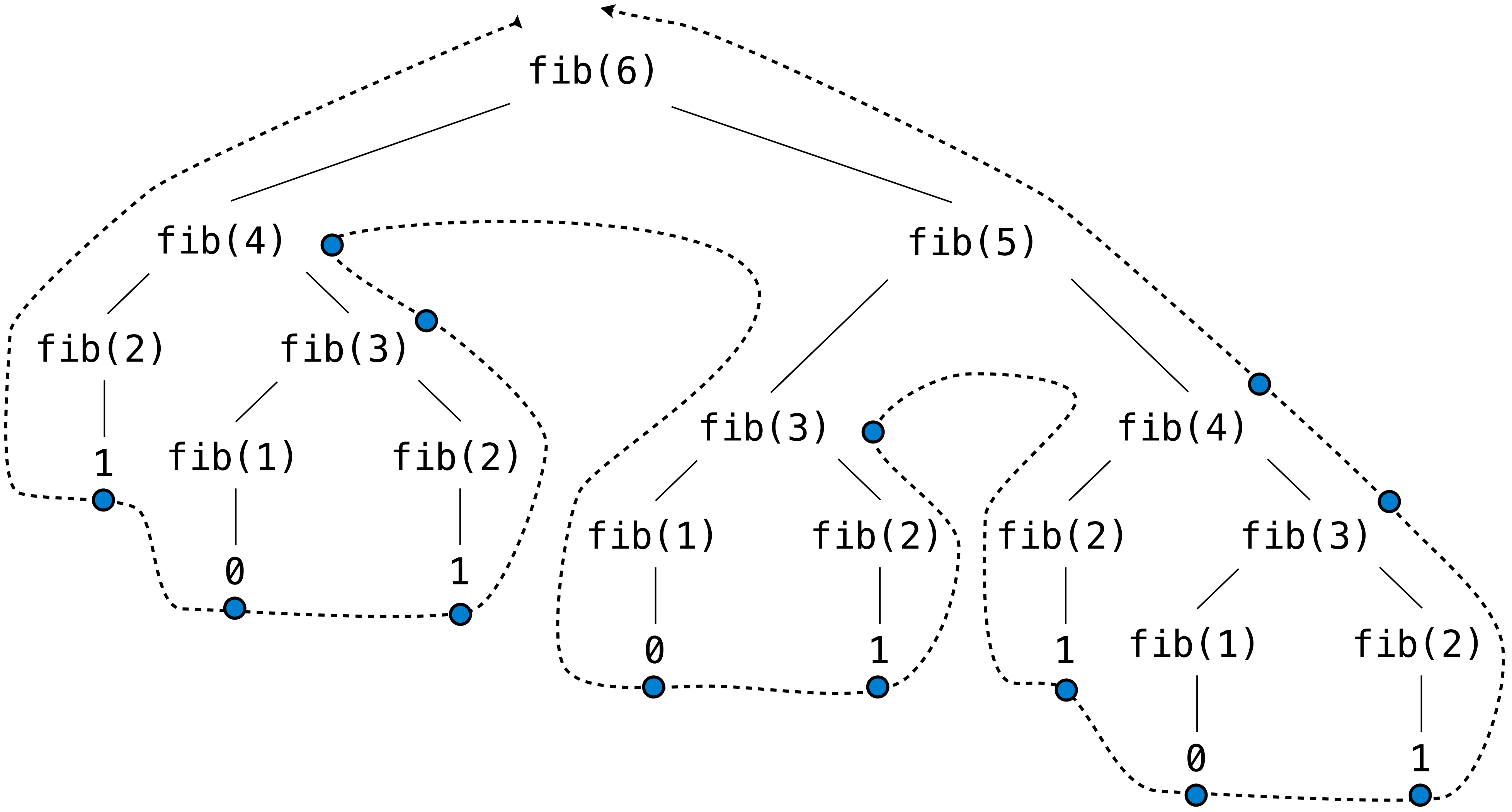

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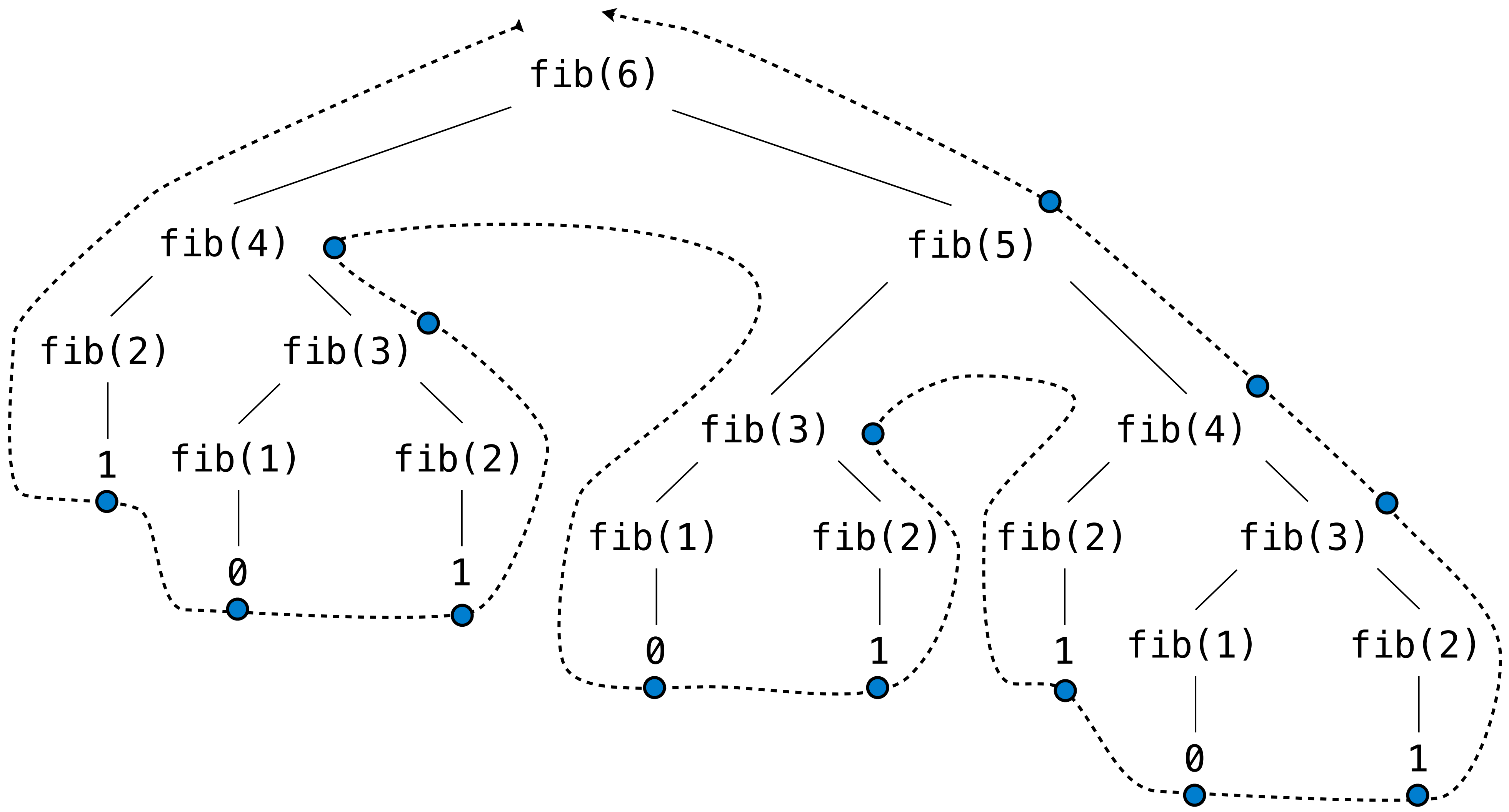

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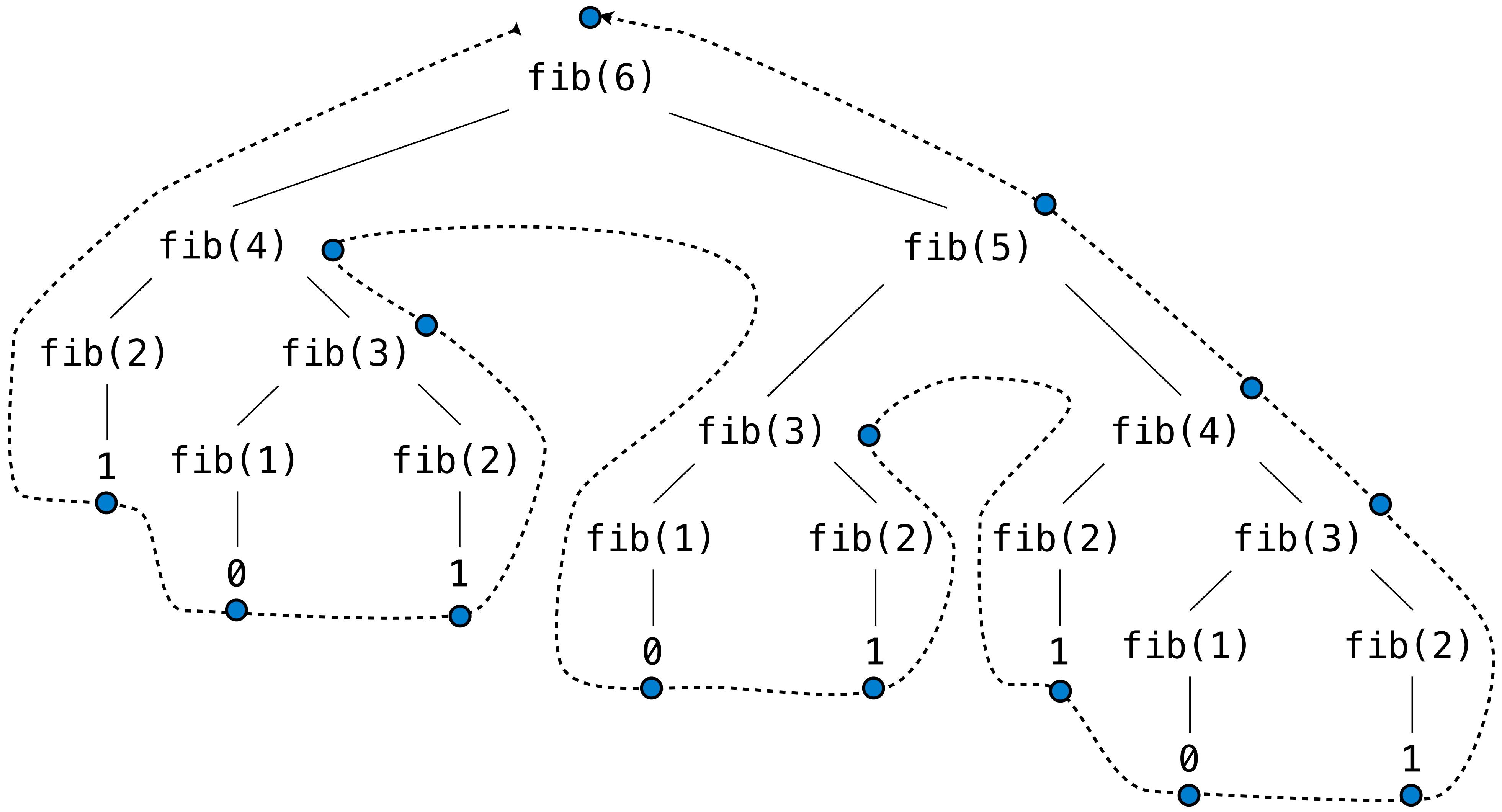

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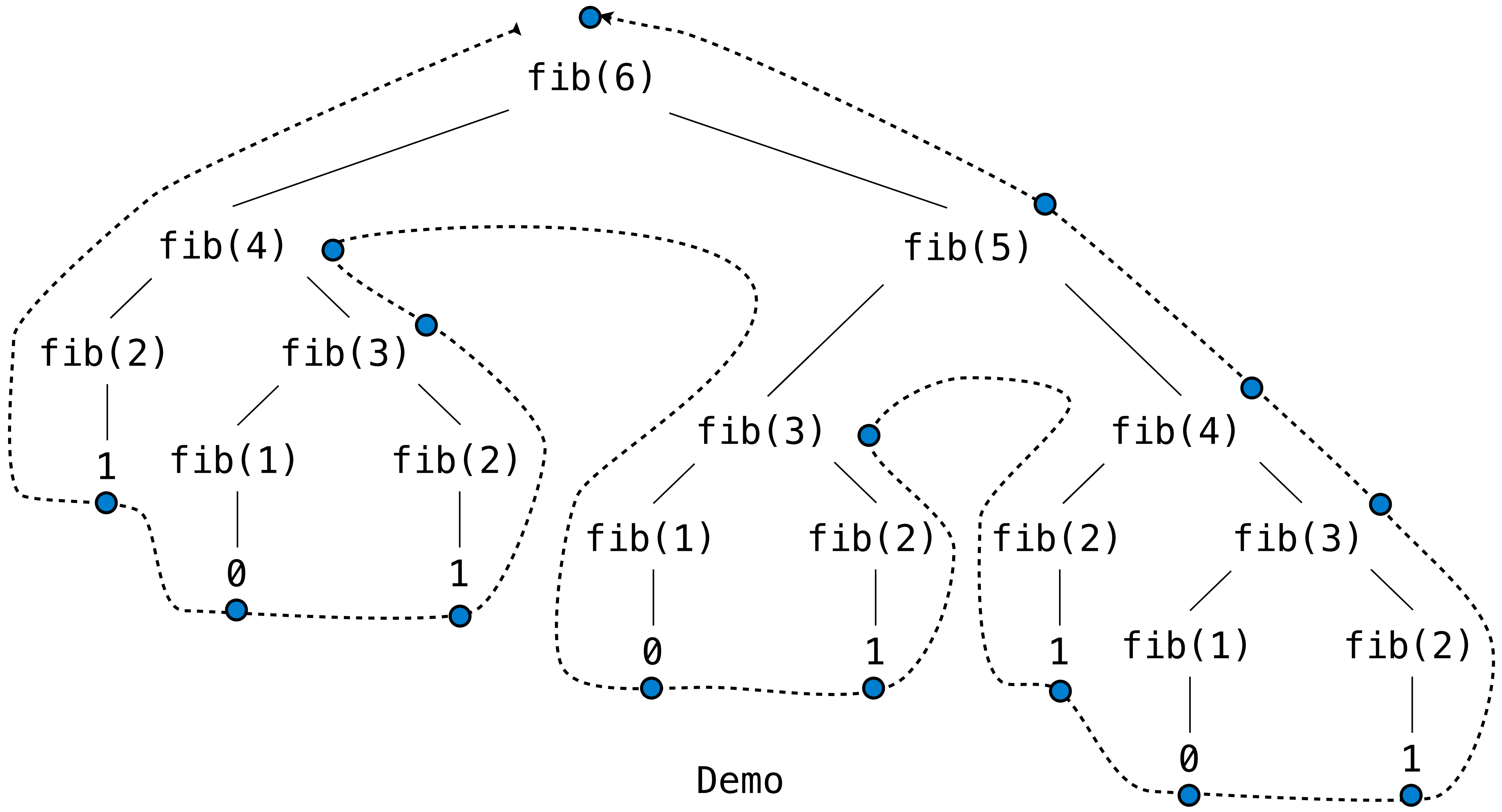

Demo

Repetition in Tree-Recursive Computation

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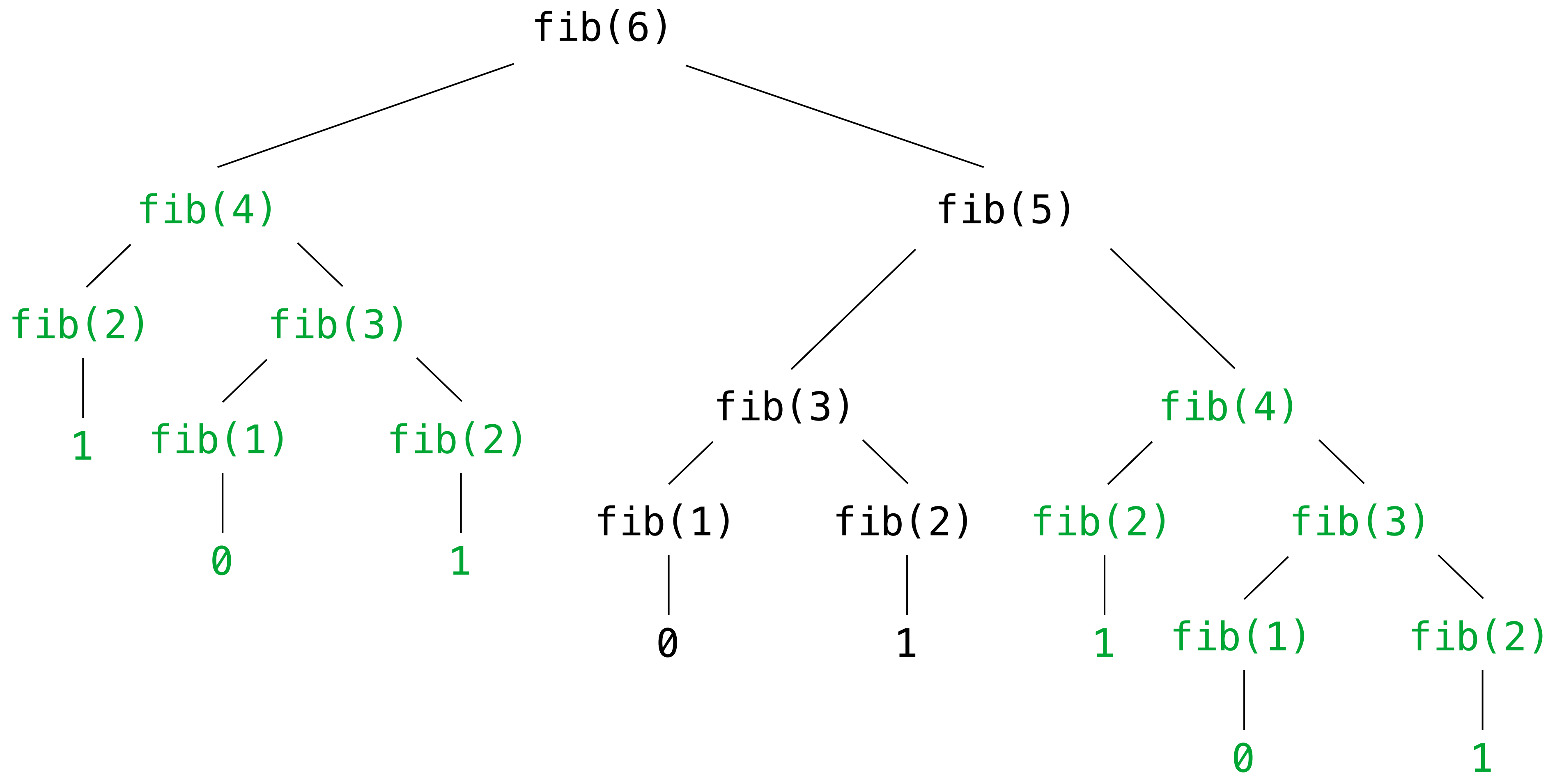

Memoization

Idea: Remember the results that have been computed before

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        if n not in cache:
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Idea: Remember the results that have been computed before

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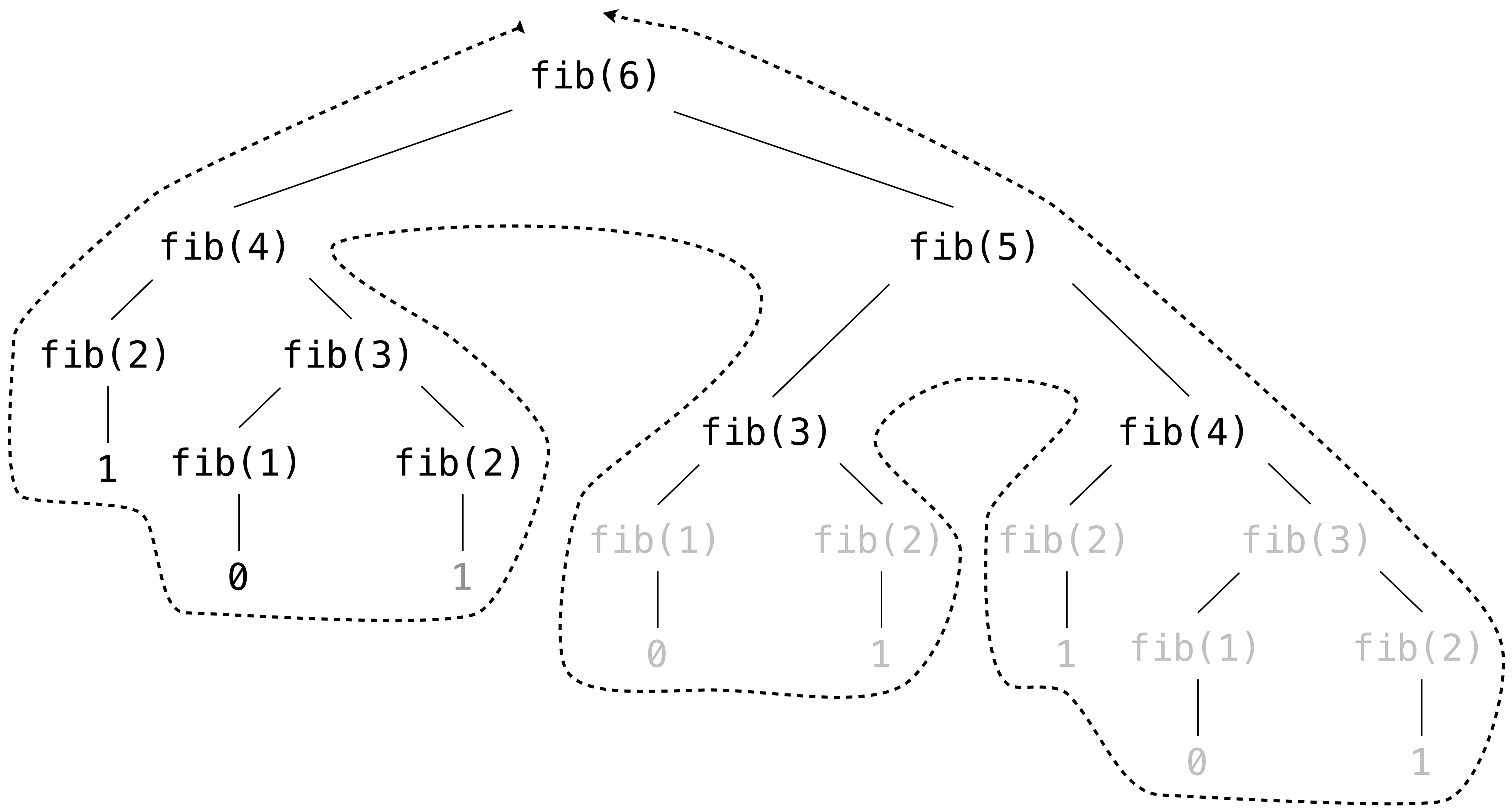
Memoized Tree Recursion



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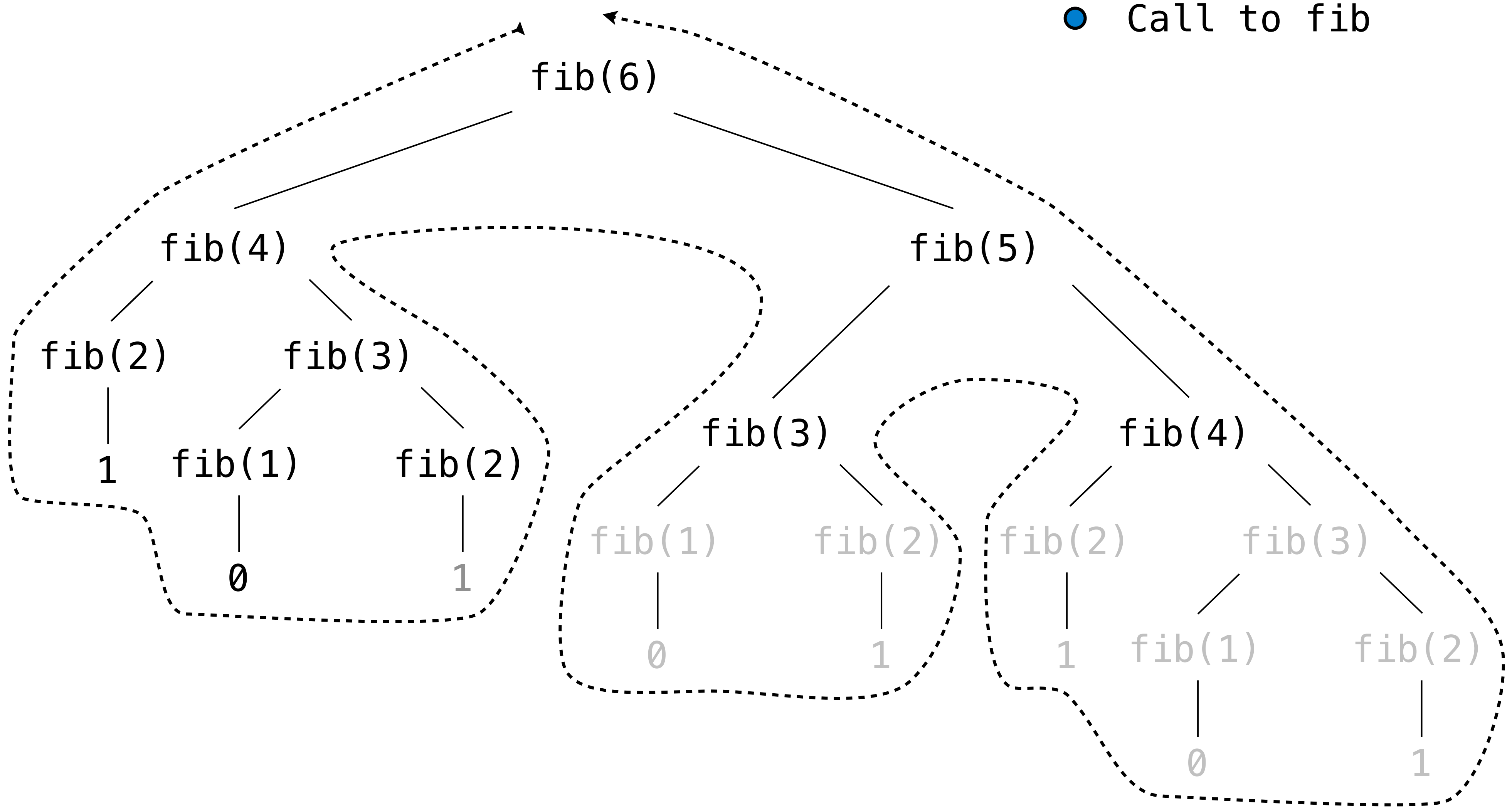

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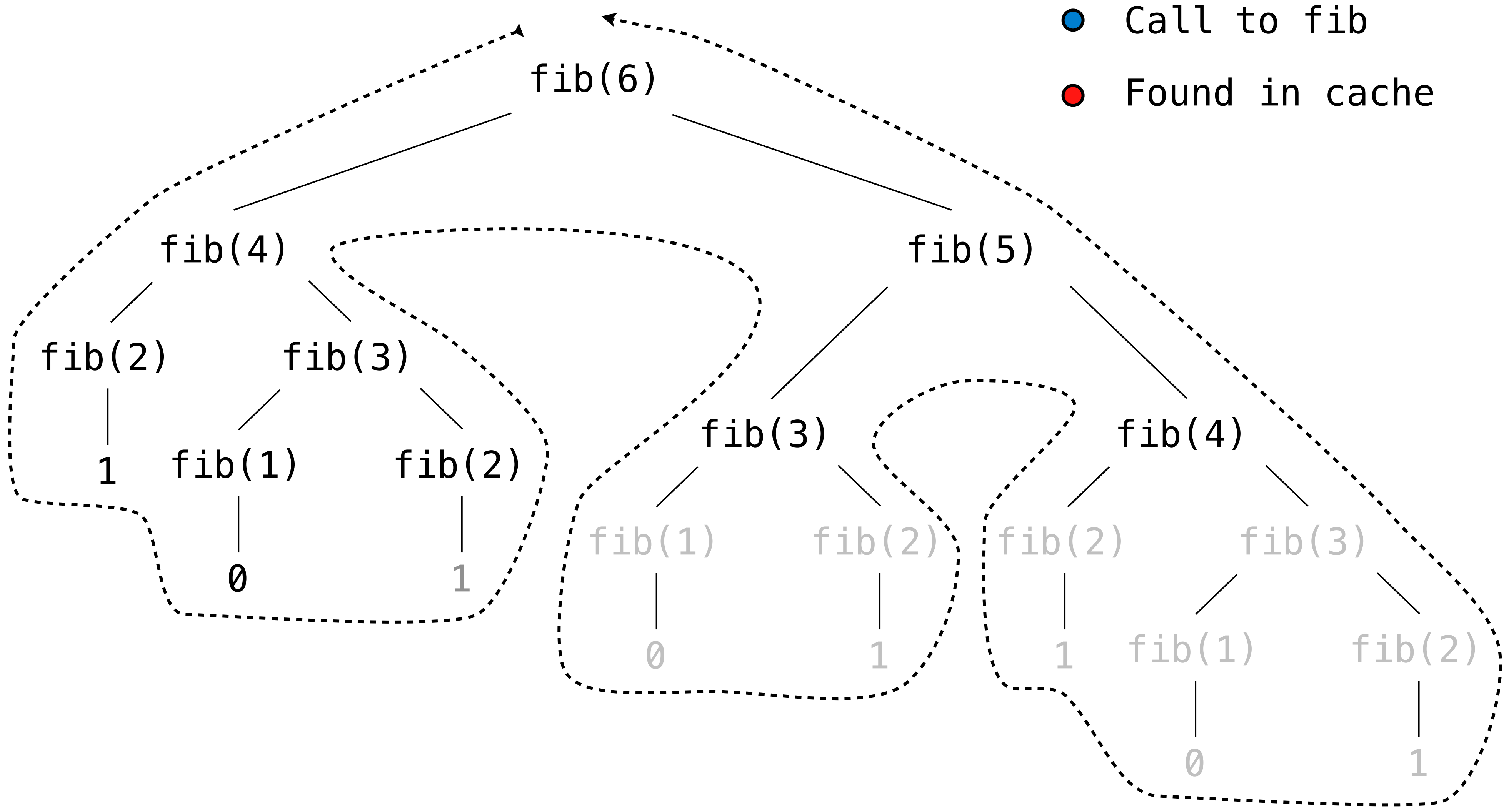

🔵 Call to fib

🔴 Found in cache

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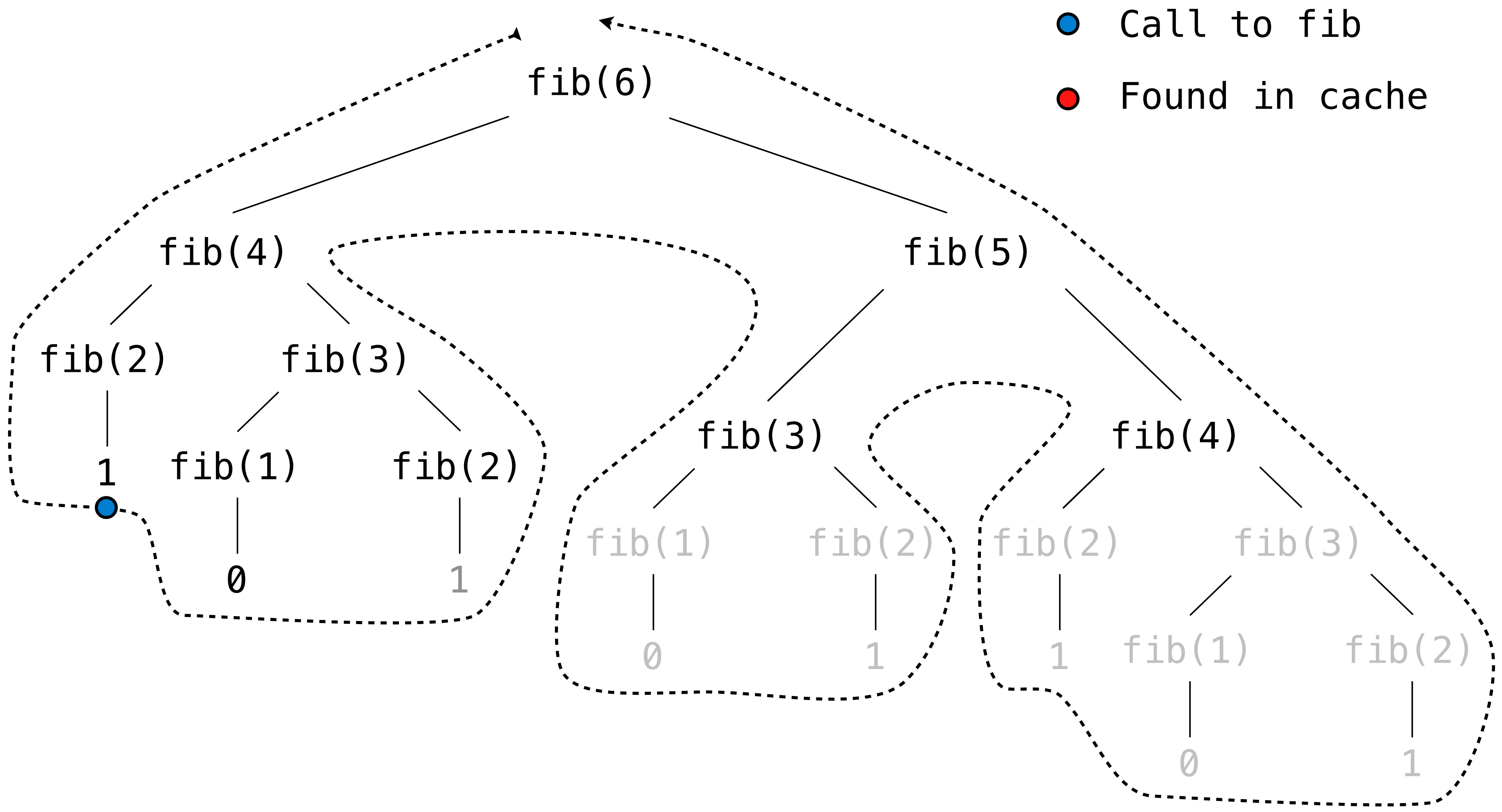

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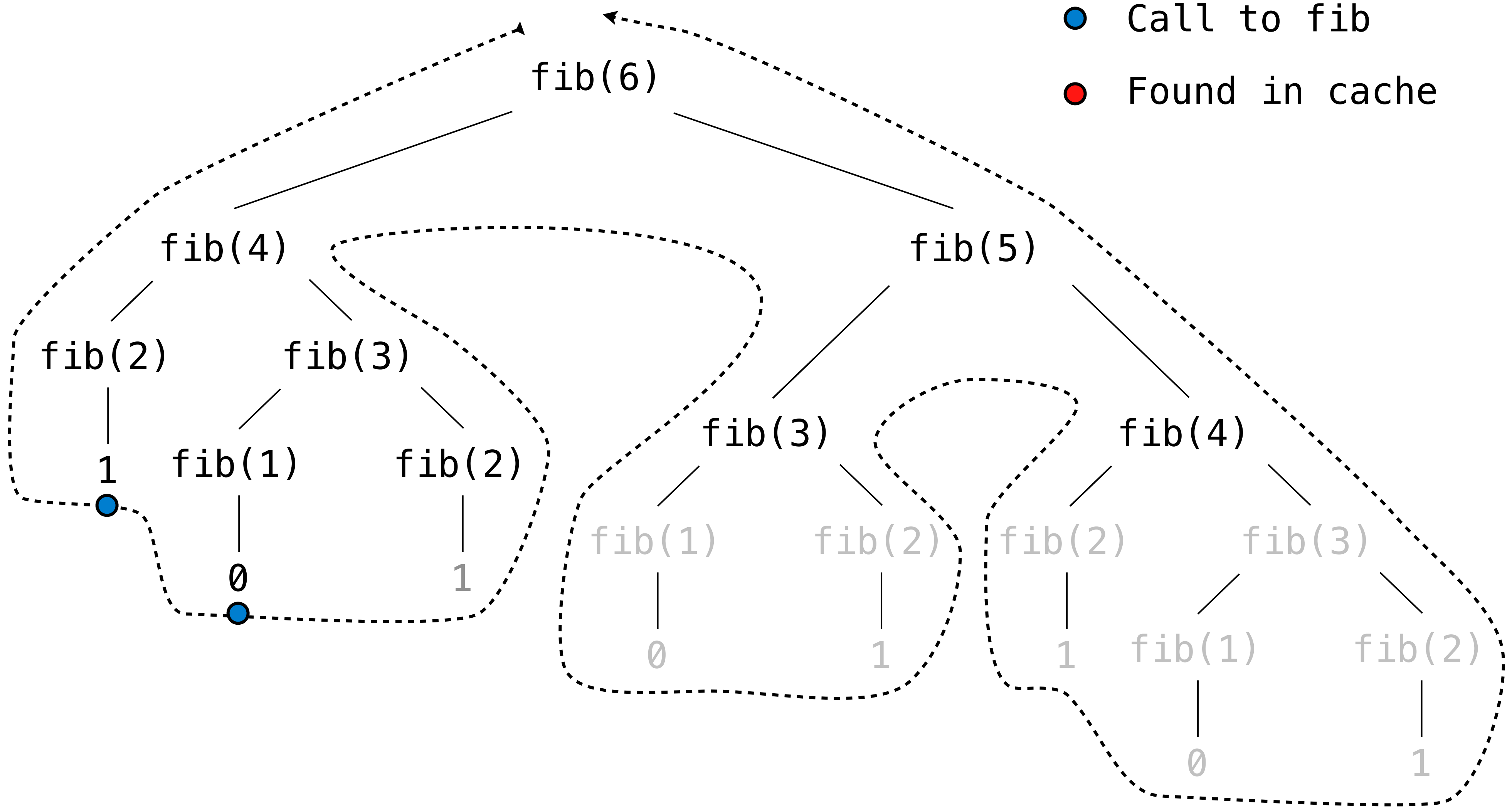

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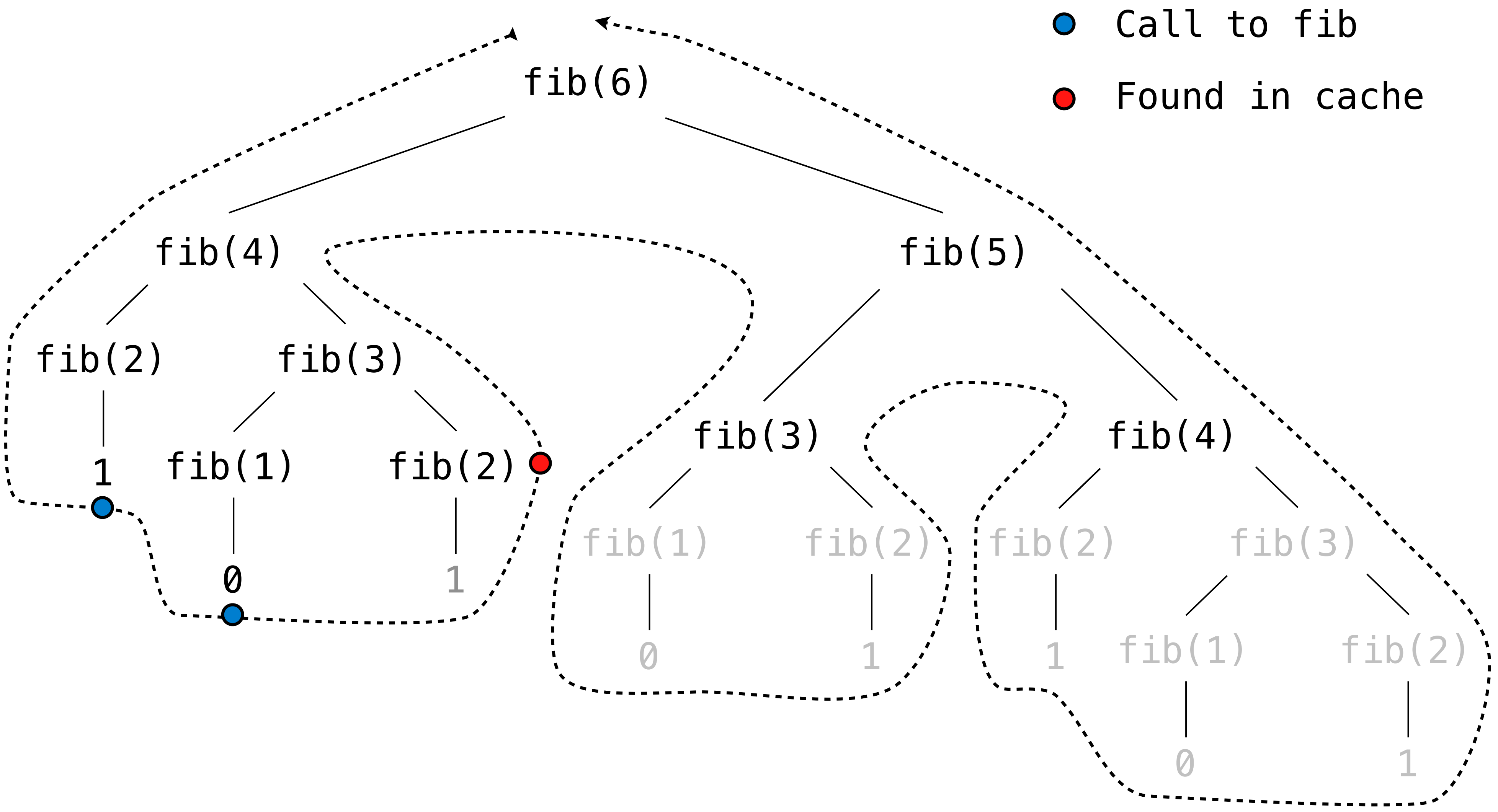

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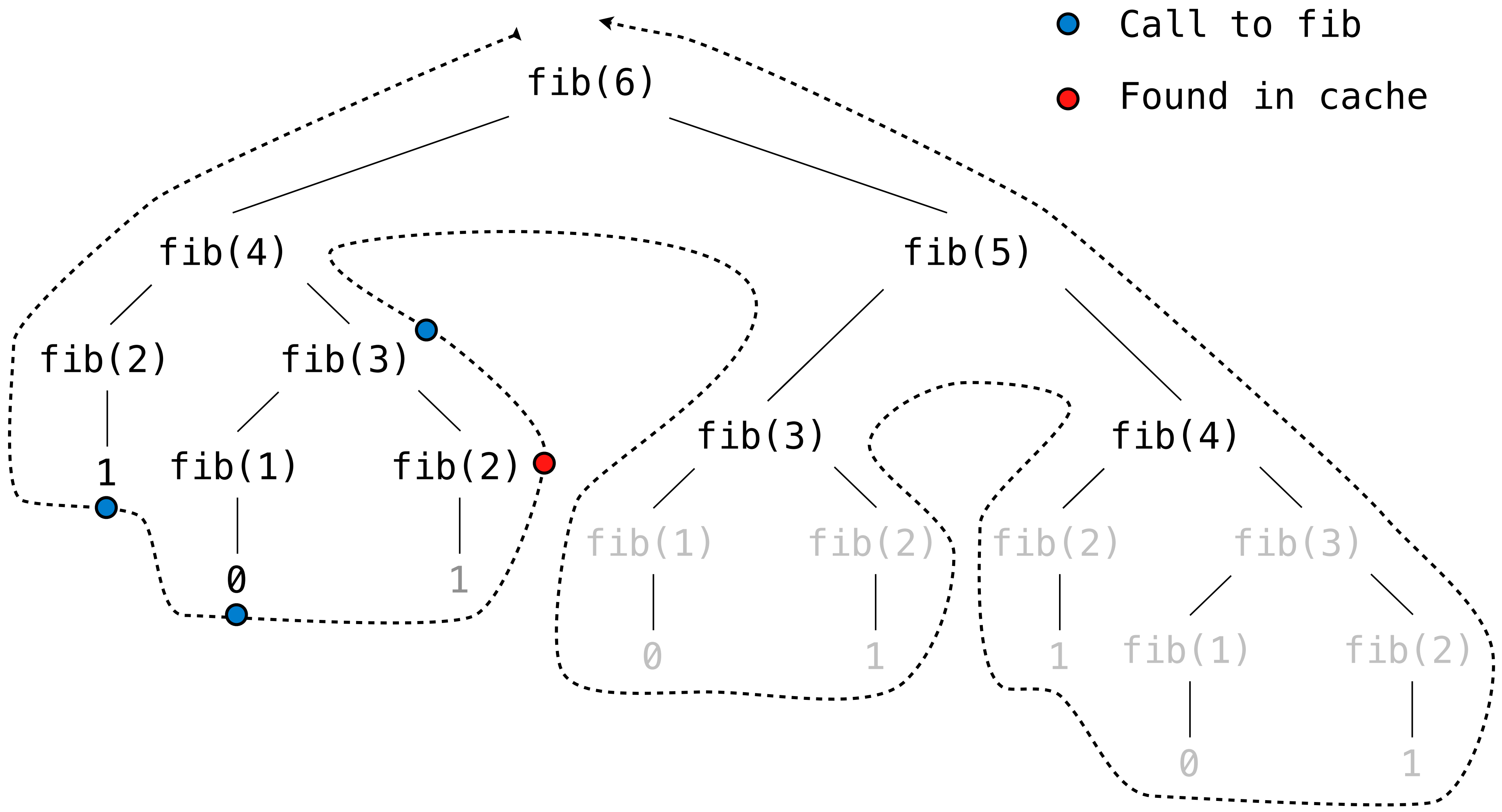

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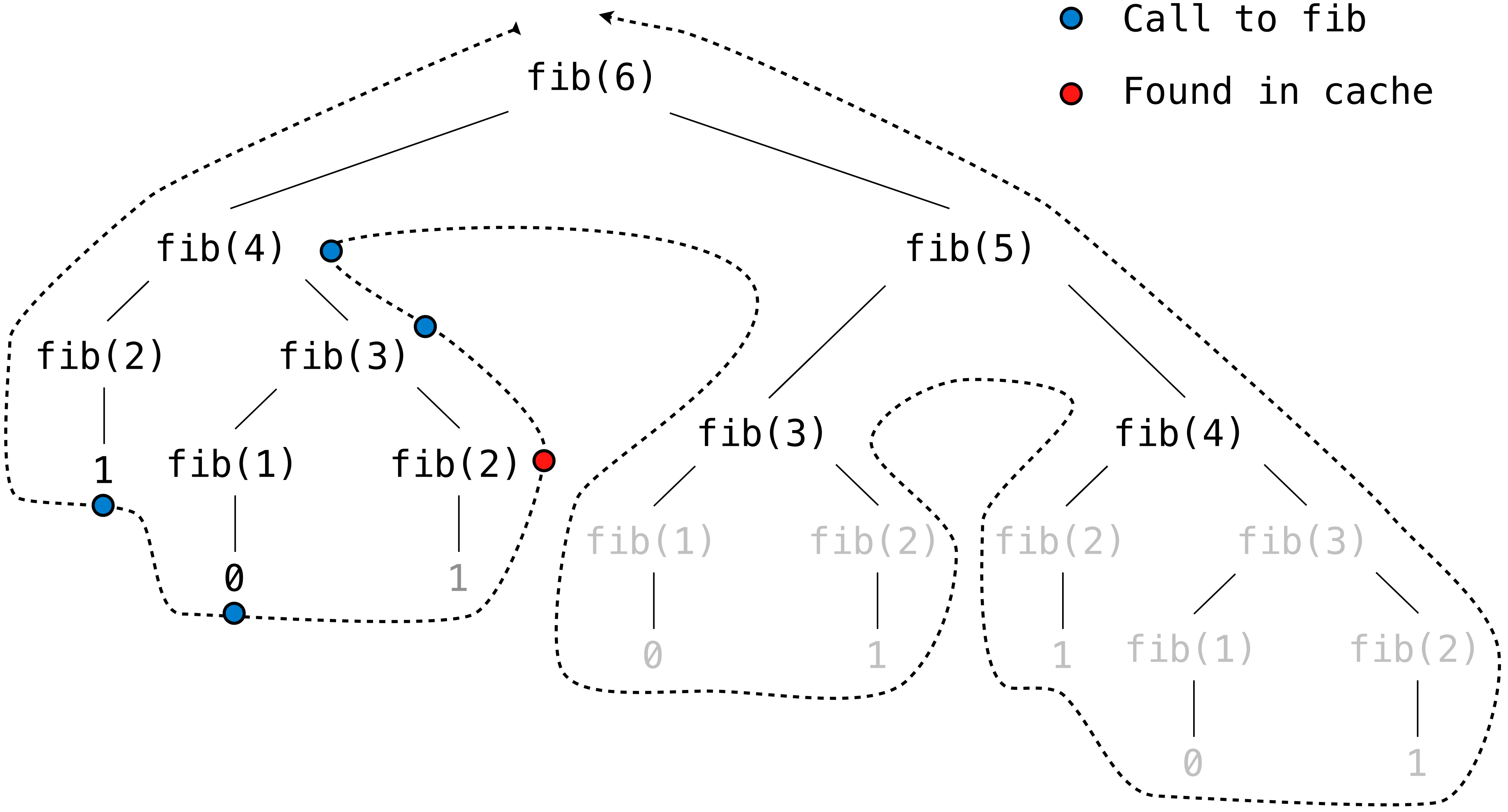

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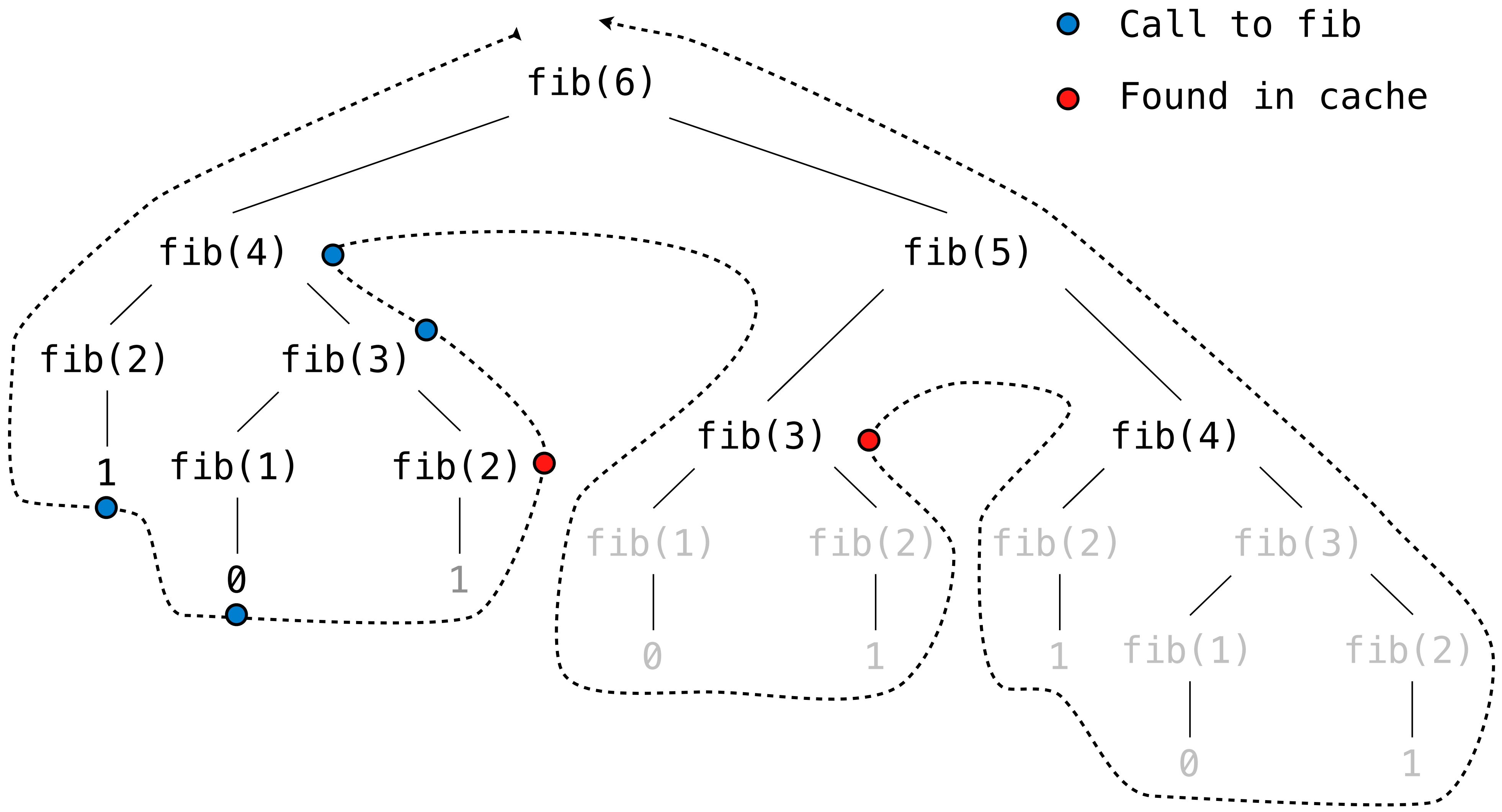

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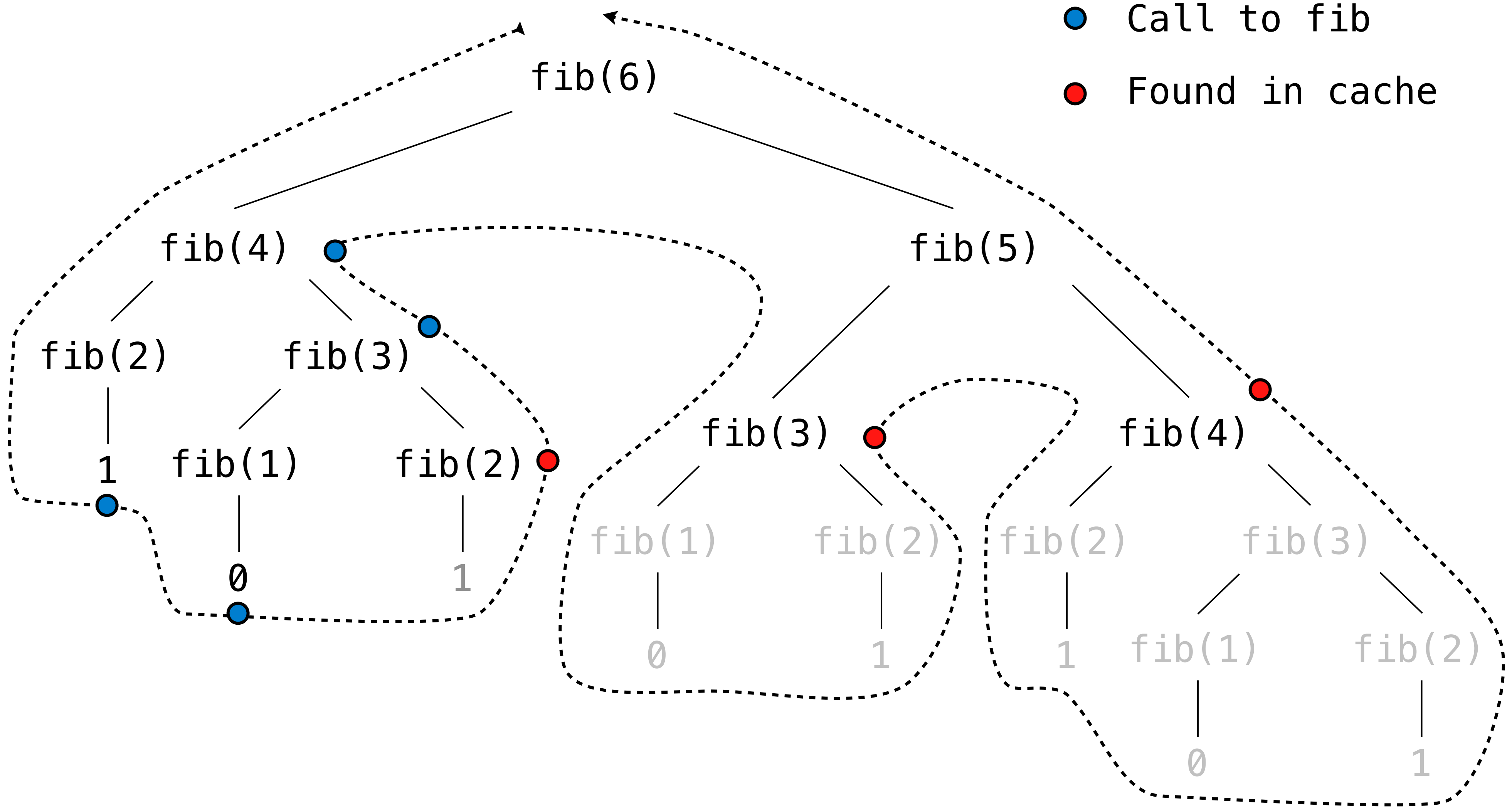

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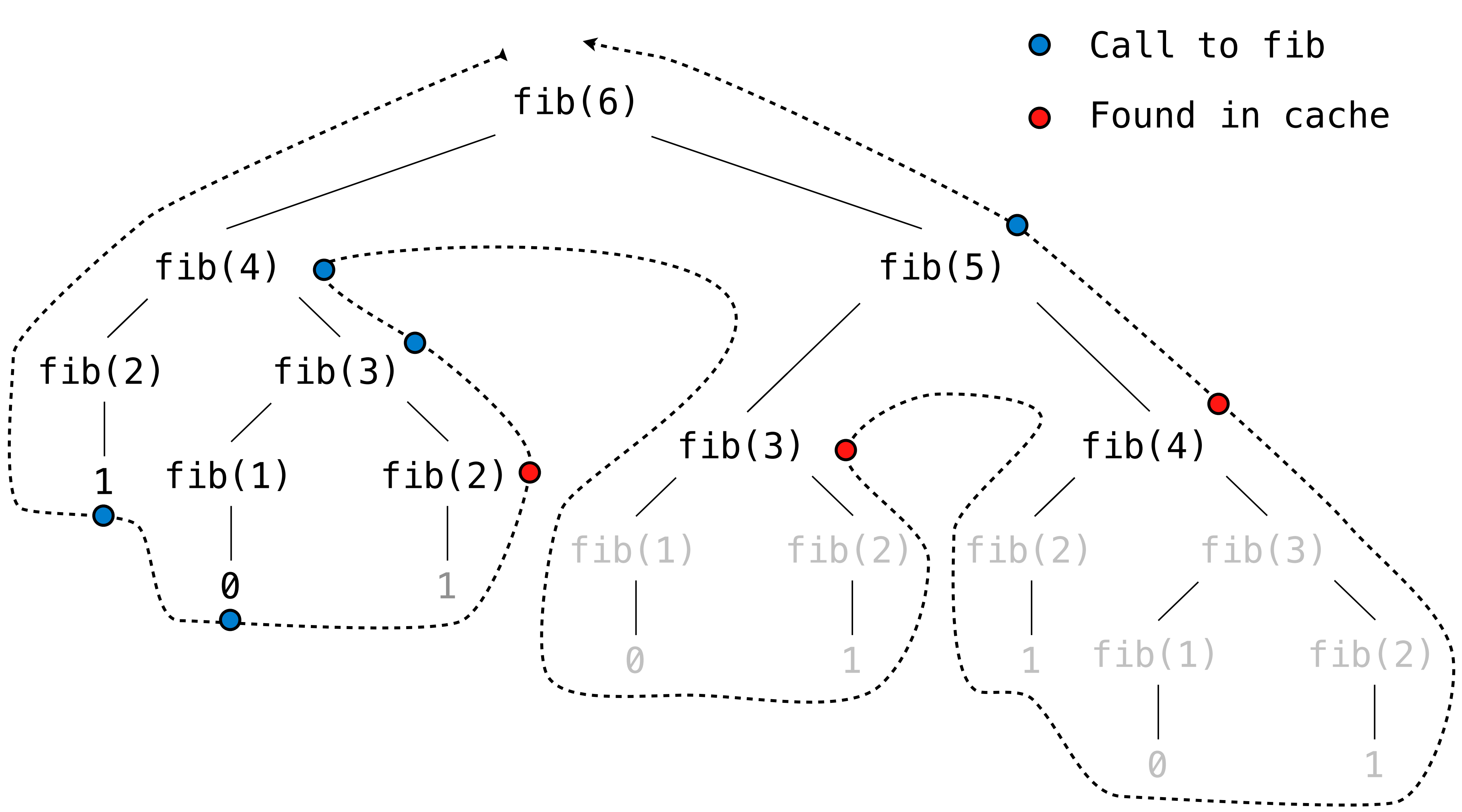

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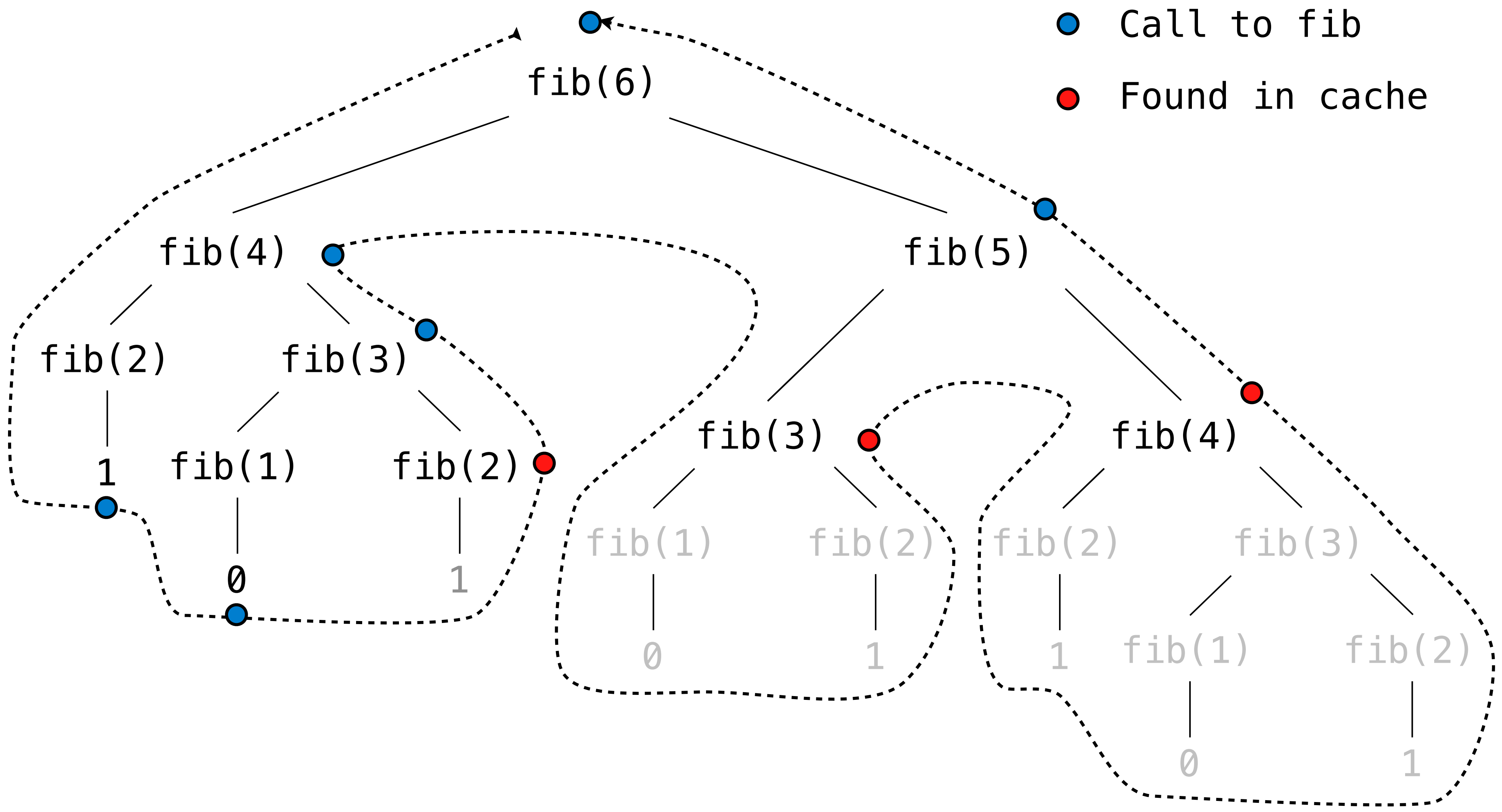

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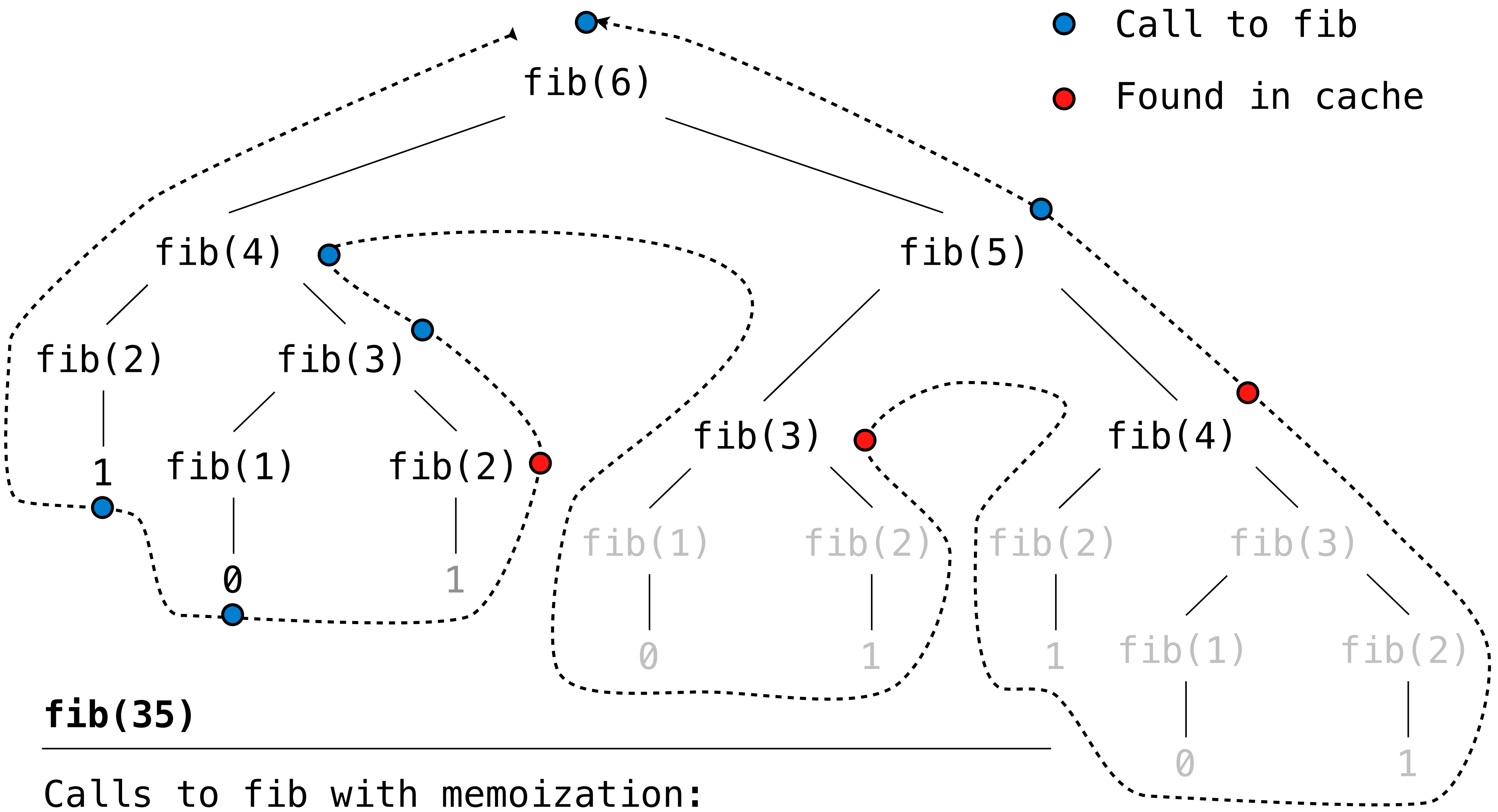

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fib(35)

Calls to fib with memoization:

Calls to fib without memoization:

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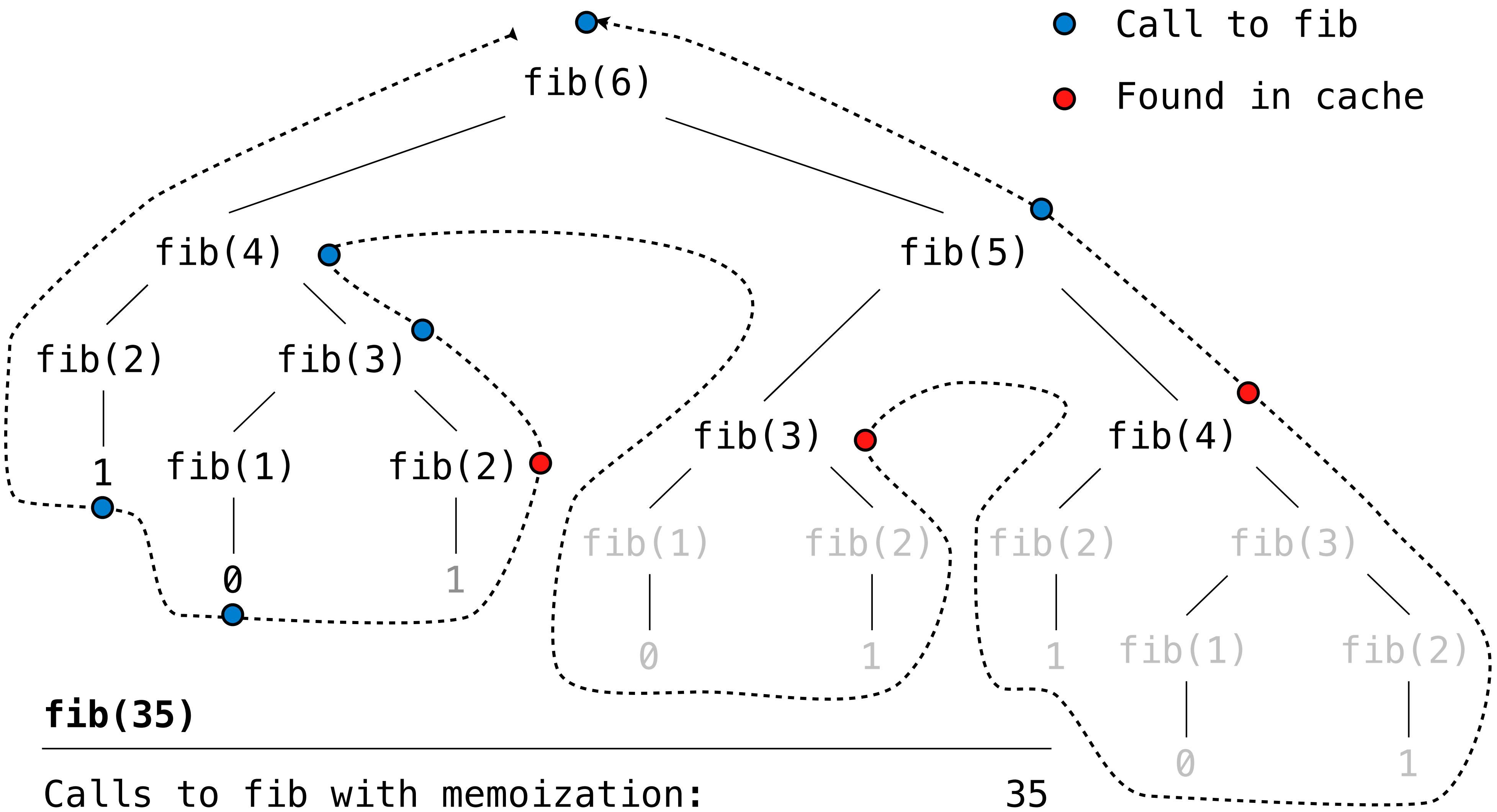

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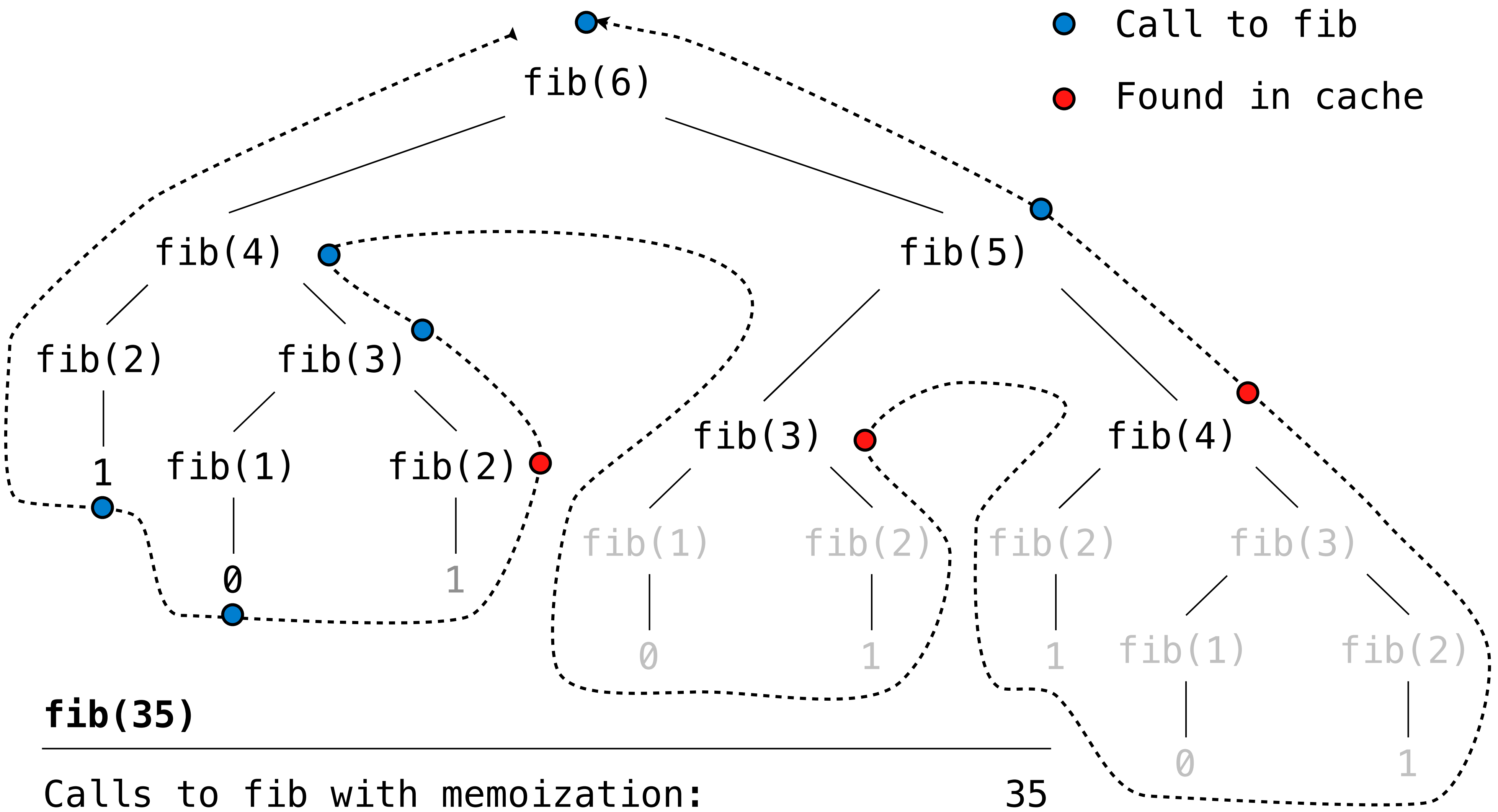

fib(35)

```
Calls to fib with memoization:              35
Calls to fib without memoization:   18,454,929
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Iteration vs Memoized Tree Recursion

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Iterative and memoized implementations are not the same.

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Time	Space
n steps	3 names
n steps	

Iteration vs Memoized Tree Recursion

Iterative and memoized implementations are not the same.

The first Fibonacci number

```
def fib_iter(n):
    prev, curr = 1, 0
    for _ in range(n-1):
        prev, curr = curr, prev + curr
    return curr
```

```
@memo
def fib(n):
    if n == 1:
        return 0
    if n == 2:
        return 1
    return fib(n-2) + fib(n-1)
```

Time	Space
n steps	3 names
n steps	n entries

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Scales with problem size

Counting Change

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$1 = $0.50 + $0.25 + $0.10 + $0.10 + $0.05
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$1 = 100 pennies
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How many ways are there to change a dollar?

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How many ways are there to change a dollar?

How many ways to change $0.11 with nickels & pennies?

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```
$0.11 can be changed with nickels & pennies by
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A. Not using any more nickels; $0.11 with just pennies

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How many ways to change $0.11 with nickels & pennies?

$0.11 can be changed with nickels & pennies by

A. Not using any more nickels; $0.11 with just pennies

B. Using at least one nickel; $0.06 with nickels & pennies

Counting Change Recursively

How many ways are there to change a dollar?

Counting Change Recursively

`How many ways are there to change a dollar?`

`The number of ways to change an amount` **`a`** `using` **`n`** `kinds =`

Counting Change Recursively

How many ways are there to change a dollar?

The number of ways to change an amount **a** using **n** kinds =

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    d = kinds[0]
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Demo