Announcements

• Homework 6 is due Monday 10/20 @ 11:59pm

• Project 3 is due Thursday 10/23 @ 11:59pm
  ▪ Project/Homework party on Monday 10/20 6pm–8pm in 155 Dwinelle

• Midterm 2 is on Monday 10/27 7pm–9pm
  ▪ Class Conflict? Fill out the conflict form at the top of http://cs61a.org
  ▪ Review session on Saturday 10/25 3pm–6pm in 2050 VLSB
Time
The Consumption of Time

Implementations of the same functional abstraction can require different amounts of time

Problem: How many factors does a positive integer \( n \) have?

A factor \( k \) of \( n \) is a positive integer that evenly divides \( n \)

```python
def factors(n):
    Slow: Test each \( k \) from 1 through \( n \)
    Fast: Test each \( k \) from 1 to square root \( n \)
          For every \( k \), \( n/k \) is also a factor!

          Time (number of divisions)

          Greatest integer less than \( \sqrt{n} \)

          (Demo)
```
Space
The Consumption of Space

Which environment frames do we need to keep during evaluation?

At any moment there is a set of active environments.

Values and frames in active environments consume memory.

Memory that is used for other values and frames can be recycled.

**Active environments:**

- Environments for any function calls currently being evaluated.
- Parent environments of functions named in active environments.

(Demo)

Interactive Diagram
Fibonacci Space Consumption

Assume we have reached this step.
Fibonacci Space Consumption

Assume we have reached this step

Has an active environment
Can be reclaimed
Hasn't yet been created
Order of Growth
Order of Growth

A method for bounding the resources used by a function by the "size" of a problem

\( n: \) size of the problem

\( R(n): \) measurement of some resource used (time or space)

\[ R(n) = \Theta(f(n)) \]

means that there are positive constants \( k_1 \) and \( k_2 \) such that

\[ k_1 \cdot f(n) \leq R(n) \leq k_2 \cdot f(n) \]

for all \( n \) larger than some minimum \( m \)
Counting Factors

Number of operations required to count the factors of n using factors_fast is $\Theta(\sqrt{n})$

To check the lower bound, we choose $k_1 = 1$:

• Statements outside the while: 4 or 5
• Statements within the while (including header): 3 or 4
• while statement iterations: between $\sqrt{n} - 1$ and $\sqrt{n}$
• Total number of statements executed: at least $4 + 3(\sqrt{n} - 1)$

To check the upper bound

• Maximum statements executed: $5 + 4\sqrt{n}$
• Maximum operations required per statement: some $p$
• We choose $k_2 = 5p$ and $m = 25$

```python
def factors_fast(n):
    sqrt_n = sqrt(n)
k, total = 1, 0
while k < sqrt_n:
    if divides(k, n):
        total += 2
    k += 1
if k * k == n:
    total += 1
return total```

Assumption: every statement, such as addition-then-assignment using the += operator, takes some fixed number of operations to execute
Order of Growth of Counting Factors

Implementations of the same functional abstraction can require different amounts of time

Problem: How many factors does a positive integer n have?

A factor k of n is a positive integer that evenly divides n

```python
def factors(n):
    Slow: Test each k from 1 through n
    Time: \( \Theta(n) \)
    Space: \( \Theta(1) \)

    Fast: Test each k from 1 to square root n
    For every k, n/k is also a factor!
    Time: \( \Theta(\sqrt{n}) \)
    Space: \( \Theta(1) \)
```

Assumption: integers occupy a fixed amount of space
Exponentiation
Exponentiation

**Goal:** one more multiplication lets us double the problem size

```python
def exp(b, n):
    if n == 0:
        return 1
    else:
        return b * exp(b, n-1)

def square(x):
    return x**x

def exp_fast(b, n):
    if n == 0:
        return 1
    elif n % 2 == 0:
        return square(exp_fast(b, n//2))
    else:
        return b * exp_fast(b, n-1)
```

\[ b^n = \begin{cases} 
1 & \text{if } n = 0 \\
 b \cdot b^{n-1} & \text{otherwise} 
\end{cases} \]

\[ b^n = \begin{cases} 
1 & \text{if } n = 0 \\
 (b^{\frac{1}{2}})^2 & \text{if } n \text{ is even} \\
 b \cdot b^{n-1} & \text{if } n \text{ is odd} 
\end{cases} \]

(Demo)
**Exponentiation**

**Goal:** one more multiplication lets us double the problem size

```python
def exp(b, n):
    if n == 0:
        return 1
    else:
        return b * exp(b, n-1)

def square(x):
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def exp_fast(b, n):
    if n == 0:
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    else:
        return b * exp_fast(b, n-1)
```

<table>
<thead>
<tr>
<th>Time</th>
<th>Space</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\Theta(n))</td>
<td>(\Theta(n))</td>
</tr>
<tr>
<td>(\Theta(\log n))</td>
<td>(\Theta(\log n))</td>
</tr>
</tbody>
</table>
Comparing Orders of Growth
Properties of Orders of Growth

**Constants:** Constant terms do not affect the order of growth of a process

\[ \Theta(n) \quad \Theta(500 \cdot n) \quad \Theta\left(\frac{1}{500} \cdot n\right) \]

**Logarithms:** The base of a logarithm does not affect the order of growth of a process

\[ \Theta(\log_2 n) \quad \Theta(\log_{10} n) \quad \Theta(\ln n) \]

**Nesting:** When an inner process is repeated for each step in an outer process, multiply the steps in the outer and inner processes to find the total number of steps

\[
\text{\begin{align*}
\text{def } \text{overlap}(a, b): \\
\text{count} &= 0 \\
\text{for } \text{item} \text{ in } a: \\
&\quad \text{if } \text{item} \text{ in } b: \\
&\quad\quad \text{count} \text{ += 1} \\
\text{return } \text{count}
\end{align*}}
\]

If \(a\) and \(b\) are both length \(n\), then \(\text{overlap}\) takes \(\Theta(n^2)\) steps
## Comparing orders of growth (n is the problem size)

<table>
<thead>
<tr>
<th>Complexity</th>
<th>Description</th>
<th>Example</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Theta(b^n)$</td>
<td>Exponential growth. Recursive <code>fib</code> takes $\Theta(\phi^n)$ steps, where $\phi = \frac{1 + \sqrt{5}}{2} \approx 1.61828$</td>
<td>Incrementing the problem scales $R(n)$ by a factor</td>
</tr>
<tr>
<td>$\Theta(n^2)$</td>
<td>Quadratic growth. E.g., <code>overlap</code></td>
<td>Incrementing $n$ increases $R(n)$ by the problem size $n$</td>
</tr>
<tr>
<td>$\Theta(n)$</td>
<td>Linear growth. E.g., slow <code>factors</code> or <code>exp</code></td>
<td></td>
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<tr>
<td>$\Theta(\sqrt{n})$</td>
<td>Square root growth. E.g., fast <code>factors</code></td>
<td></td>
</tr>
<tr>
<td>$\Theta(\log n)$</td>
<td>Logarithmic growth. E.g., fast <code>exp</code></td>
<td>Doubling the problem only increments $R(n)$</td>
</tr>
<tr>
<td>$\Theta(1)$</td>
<td>Constant. The problem size doesn't matter</td>
<td></td>
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