Announcements
Measuring Efficiency
Recursive Computation of the Fibonacci Sequence

Our first example of tree recursion:
Recursive Computation of the Fibonacci Sequence

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```python
def fib(n):
    if n <= 1:
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Our first example of tree recursion:

\[
def \text{fib}(n):
    \text{if } n \leq 1:\n        \text{return } 0
    \text{elif } n == 1:\n        \text{return } 1
    \text{else:}\n        \text{return } \text{fib}(n-2) + \text{fib}(n-1)
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[Image: Recursive Computation of the Fibonacci Sequence Diagram]
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Memoization
Memoization

Idea: Remember the results that have been computed before
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def memo(f):

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```python
def memo(f):
    cache = {}
```
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**Idea:** Remember the results that have been computed before

```python
def memo(f):
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    def memoized(n):
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def memo(f):
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        if n not in cache:
```
Memoization

**Idea:** Remember the results that have been computed before

```python
def memo(f):
    cache = {}
    def memoized(n):
        if n not in cache:
            cache[n] = f(n)
```

Memoization

Idea: Remember the results that have been computed before

def memo(f):
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```python
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    cache = {}
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            return cache[n]
        return memoized
    return memoized
```
Memoization

**Idea:** Remember the results that have been computed before

```python
def memo(f):
    cache = {}
    def memoized(n):
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    return memoized
```

Keys are arguments that map to return values.
**Memoization**

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        if n not in cache:
            cache[n] = f(n)
        return cache[n]
    return memoized
```

Keys are arguments that map to return values

Same behavior as f, if f is a pure function
Memoization

**Idea:** Remember the results that have been computed before

```python
def memo(f):
    cache = {}  # Keys are arguments that map to return values

def memoized(n):
    if n not in cache:
        cache[n] = f(n)
    return cache[n]

return memoized  # Same behavior as f, if f is a pure function
```

(Demo)
Memoized Tree Recursion

```
fib(5)
  fib(3)
  fib(1)  fib(2)
  1  fib(0)  fib(1)
  0  1

fib(4)
  fib(2)
  fib(0)  fib(1)
  0  1

fib(3)
  fib(1)  fib(2)
  fib(0)  fib(1)
  0  1

fib(2)
  fib(0)  fib(1)
  0  1

fib(1)
  fib(0)  fib(1)
  0  1

fib(0)
```
Memoized Tree Recursion

Call to `fib`

```
Call to `fib`

```

```
fib(5)

```

```
fib(3)

```

```
fib(1)  fib(2)
1  fib(0)  fib(1)
0  1

```

```
fib(4)

```

```
fib(2)

```

```
fib(0)  fib(1)
0  1

```

```
fib(3)

```

```
fib(1)  fib(2)
1  fib(0)  fib(1)
0  1

```

```
fib(0)  fib(1)
0  1

```

```
```
Memoized Tree Recursion

Call to \textit{fib}  

\textcolor{red}{\text{Found in cache}}
Memoized Tree Recursion

Call to fib
- Found in cache
- Skipped
Memoized Tree Recursion

Call to fib
- Found in cache
- Skipped
Memoized Tree Recursion

Call to fib
- Found in cache
- Skipped
Memoized Tree Recursion

Call to fib

Found in cache

Skipped
Memoized Tree Recursion

Call to fib
- Found in cache
- Skipped
Memoized Tree Recursion

Call to fib
- Found in cache
- Skipped
Memoized Tree Recursion

- Call to fib
- Found in cache
- Skipped
Memoized Tree Recursion

- Call to fib
- Found in cache
- Skipped

Diagram showing the memoized tree recursion for calculating Fibonacci numbers.
Memoized Tree Recursion

Call to fib
- Found in cache
- Skipped

Diagram showing recursive calls and cache hits for calculating Fibonacci numbers.

- fib(5) called.
- fib(3) and fib(2) called.
- fib(0) is found in cache, fib(1) is calculated.
- fib(4) called.
- fib(2) and fib(3) called.
- fib(0) is found in cache, fib(1) is calculated.
- fib(1) is found in cache, fib(2) is calculated.
- fib(3) called.
- fib(1) found in cache, fib(2) is calculated.
- fib(2) called.
- fib(0) found in cache, fib(1) is calculated.
- fib(1) found in cache, fib(2) is calculated.
- fib(4) called.
- fib(3) found in cache, fib(1) is calculated.
- fib(2) called.
- fib(0) found in cache, fib(1) is calculated.
- fib(1) found in cache, fib(2) is calculated.
- fib(0) found in cache, fib(1) is calculated.
- fib(1) found in cache, fib(2) is calculated.
- fib(2) found in cache, fib(1) is calculated.
- fib(1) found in cache, fib(2) is calculated.
- fib(0) found in cache, fib(1) is calculated.
- fib(1) found in cache, fib(2) is calculated.
- fib(2) found in cache, fib(1) is calculated.
- fib(1) found in cache, fib(2) is calculated.
Memoized Tree Recursion

Call to fib
- Found in cache
- Skipped
Memoized Tree Recursion

- Call to fib
- Found in cache
- Skipped
Memoized Tree Recursion

```
Call to fib
Found in cache
Skipped
```
Memoized Tree Recursion

Call to $fib$
- Found in cache
- Skipped
Memoized Tree Recursion

Call to fib
- Found in cache
- Skipped
Memoized Tree Recursion

Call to fib
- Found in cache
- Skipped

fib(5)
- fib(3)
  - fib(1)
    - fib(0)
      - 0
      - 1
    - fib(1)
      - 1
  - fib(2)
    - fib(0)
      - 0
      - 1
    - fib(1)
      - 0
      - 1
- fib(4)
  - fib(2)
    - fib(0)
      - 0
      - 1
    - fib(1)
      - 1
  - fib(3)
    - fib(0)
      - 0
      - 1
    - fib(1)
      - 1
    - fib(2)
      - fib(0)
        - 0
        - 1
      - fib(1)
        - 0
        - 1
Memoized Tree Recursion

Call to fib
- Found in cache
- Skipped
The Consumption of Space
The Consumption of Space

Which environment frames do we need to keep during evaluation?
The Consumption of Space

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At any moment there is a set of active environments
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Values and frames in active environments consume memory
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Memory that is used for other values and frames can be recycled.
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Active environments:
The Consumption of Space

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Active environments:

• Environments for any function calls currently being evaluated.
The Consumption of Space

Which environment frames do we need to keep during evaluation?

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Active environments:

• Environments for any function calls currently being evaluated

• Parent environments of functions named in active environments
The Consumption of Space

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Values and frames in active environments consume memory

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Active environments:

- Environments for any function calls currently being evaluated
- Parent environments of functions named in active environments

(Demo)

Interactive Diagram
Fibonacci Space Consumption
Fibonacci Space Consumption

\[ \text{fib}(5) \]
Fibonacci Space Consumption

\[ \text{fib}(5) \]

\[ \text{fib}(3) \]
Fibonacci Space Consumption

\[ \text{fib}(5) \]

\[ \text{fib}(3) \quad \text{fib}(4) \]
Fibonacci Space Consumption

```
<table>
<thead>
<tr>
<th>fib(5)</th>
</tr>
</thead>
<tbody>
<tr>
<td>fib(3)</td>
</tr>
<tr>
<td>fib(1)</td>
</tr>
<tr>
<td>1</td>
</tr>
<tr>
<td>fib(0)</td>
</tr>
<tr>
<td>0</td>
</tr>
<tr>
<td>fib(2)</td>
</tr>
<tr>
<td>fib(1)</td>
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Fibonacci Space Consumption

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<tr>
<td></td>
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Fibonacci Space Consumption

Assume we have reached this step.
Fibonacci Space Consumption

Assume we have reached this step.
Fibonacci Space Consumption

Assume we have reached this step

fib(5)

fib(3)
  fib(1)  fib(2)
    1  fib(0) fib(1)
    0  1

fib(4)
  fib(2)
    fib(0) fib(1)
      0  1

fib(1)
  fib(2)
    fib(0) fib(1)
      0  1

fib(0) fib(1)
  1 1

Has an active environment
Fibonacci Space Consumption

Has an active environment
Can be reclaimed

Assume we have reached this step
Fibonacci Space Consumption

Assume we have reached this step

Has an active environment
Can be reclaimed
Hasn't yet been created
Time
Comparing Implementations
Comparing Implementations

Implementations of the same functional abstraction can require different resources.
Comparing Implementations

Implementations of the same functional abstraction can require different resources

Problem: How many factors does a positive integer $n$ have?
Comparing Implementations

Implementations of the same functional abstraction can require different resources

**Problem:** How many factors does a positive integer $n$ have?

A factor $k$ of $n$ is a positive integer that evenly divides $n$
Comparing Implementations

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```
Implementations of the same functional abstraction can require different resources

**Problem:** How many factors does a positive integer $n$ have?

A factor $k$ of $n$ is a positive integer that evenly divides $n$

```python
def factors(n):
    # Slow: Test each k from 1 through n
```
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**Problem:** How many factors does a positive integer \( n \) have?

A factor \( k \) of \( n \) is a positive integer that evenly divides \( n \)

```python
def factors(n):

    **Slow:** Test each \( k \) from 1 through \( n \)

    **Fast:** Test each \( k \) from 1 to square root \( n \)
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**Question:** How many time does each implementation use division? (Demo)
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def factors(n):
    # Time (number of divisions)

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Implementations of the same functional abstraction can require different resources.

**Problem:** How many factors does a positive integer \( n \) have?

A factor \( k \) of \( n \) is a positive integer that evenly divides \( n \)

```python
def factors(n):
    # Time (number of divisions)
    n
    Greatest integer less than \( \sqrt{n} \)

    **Slow:** Test each \( k \) from 1 through \( n \)

    **Fast:** Test each \( k \) from 1 to square root \( n \)
    For every \( k \), \( n/k \) is also a factor!

**Question:** How many time does each implementation use division? (Demo)```
Orders of Growth
Order of Growth

A method for bounding the resources used by a function by the "size" of a problem
Order of Growth

A method for bounding the resources used by a function by the "size" of a problem

n: size of the problem
Order of Growth

A method for bounding the resources used by a function by the "size" of a problem

\[ n: \quad \text{size of the problem} \]

\[ R(n): \quad \text{measurement of some resource used (time or space)} \]
Order of Growth

A method for bounding the resources used by a function by the "size" of a problem

\[ R(n) = \Theta(f(n)) \]

**n:** size of the problem

**R(n):** measurement of some resource used (time or space)
Order of Growth

A method for bounding the resources used by a function by the "size" of a problem

\( n: \) size of the problem

\( R(n): \) measurement of some resource used (time or space)

\[ R(n) = \Theta(f(n)) \]

means that there are positive constants \( k_1 \) and \( k_2 \) such that
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\[ k_1 \cdot f(n) \leq R(n) \leq k_2 \cdot f(n) \]
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for all \( n \) larger than some minimum \( m \)
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\( R(n): \) measurement of some resource used (time or space)

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Order of Growth of Counting Factors

Implementations of the same functional abstraction can require different amounts of time

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    Time  Space

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<table>
<thead>
<tr>
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<th>Space</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \Theta(n) )</td>
<td>( \Theta(1) )</td>
</tr>
</tbody>
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Order of Growth of Counting Factors

Implementations of the same functional abstraction can require different amounts of time

Problem: How many factors does a positive integer $n$ have?

A factor $k$ of $n$ is a positive integer that evenly divides $n$

```python
def factors(n):
    # Slow: Test each $k$ from 1 through $n$
    # Time: $\Theta(n)$, Space: $\Theta(1)$
    # Fast: Test each $k$ from 1 to square root $n$
    # For every $k$, $n/k$ is also a factor!
    # Time: $\Theta(\sqrt{n})$, Space: $\Theta(1)$
```

Order of Growth of Counting Factors

Implementations of the same functional abstraction can require different amounts of time

Problem: How many factors does a positive integer n have?

A factor k of n is a positive integer that evenly divides n

def factors(n):

    Slow: Test each k from 1 through n
          \[ \Theta(n) \] \[ \Theta(1) \]  

    Fast: Test each k from 1 to square root n
          For every k, n/k is also a factor!
          \[ \Theta(\sqrt{n}) \] \[ \Theta(1) \]
Order of Growth of Counting Factors

Implementations of the same functional abstraction can require different amounts of time.

**Problem:** How many factors does a positive integer $n$ have?

A factor $k$ of $n$ is a positive integer that evenly divides $n$.

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def factors(n):
    
    **Slow:** Test each $k$ from 1 through $n$
    
    **Fast:** Test each $k$ from 1 to square root $n$
    For every $k$, $n/k$ is also a factor!

    | Time    | Space |
    |---------|-------|
    | $\Theta(n)$ | $\Theta(1)$ |
    | $\Theta(\sqrt{n})$ | $\Theta(1)$ |
```

(Demo)

Assumption: integers occupy a fixed amount of space.
Exponentiation
Exponentiation
Exponentiation

**Goal:** one more multiplication lets us double the problem size
Exponentiation

**Goal:** one more multiplication lets us double the problem size

```python
def exp(b, n):
    if n == 0:
        return 1
    else:
        return b * exp(b, n-1)
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**Exponentiation**

**Goal:** one more multiplication lets us double the problem size

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\[ b^n = \begin{cases} 
1 & \text{if } n = 0 \\
 b \cdot b^{n-1} & \text{otherwise} 
\end{cases} \]
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b^n = \begin{cases} 
1 & \text{if } n = 0 \\
(b^{\frac{1}{2}n})^2 & \text{if } n \text{ is even} \\
\cdot b^{n-1} & \text{if } n \text{ is odd}
\end{cases}
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def square(x):
    return x**x

def exp_fast(b, n):
    if n == 0:
        return 1
    elif n % 2 == 0:
        return square(exp_fast(b, n//2))
    else:
        return b * exp_fast(b, n-1)
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<table>
<thead>
<tr>
<th>Time</th>
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<tbody>
<tr>
<td>(n)</td>
<td>(\Theta(n))</td>
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def square(x):
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```
Comparing Orders of Growth
Properties of Orders of Growth
Properties of Orders of Growth

**Constants:** Constant terms do not affect the order of growth of a process.
Properties of Orders of Growth

**Constants:** Constant terms do not affect the order of growth of a process.

\[ \Theta(n) \]
Properties of Orders of Growth

Constants: Constant terms do not affect the order of growth of a process

$\Theta(n)$  $\Theta(500 \cdot n)$
Properties of Orders of Growth

**Constants:** Constant terms do not affect the order of growth of a process

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def overlap(a, b):
    count = 0
    for item in a:
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            count += 1
    return count
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*Outer: length of a\n Inner: length of b*
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def overlap(a, b):
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If \( a \) and \( b \) are both length \( n \), then overlap takes \( \Theta(n^2) \) steps
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\[
\text{steps in outer process} \times \text{steps in inner process} = \left(n \times 500 \times \frac{1}{500} \times \log_2 n \times \log_{10} n \times \ln n\right)
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If a and b are both length \( n \), then overlap takes \( \Theta(n^2) \) steps.

**Lower-order terms:** The fastest-growing part of the computation dominates the total.
Properties of Orders of Growth

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\[
\left( n \right) \times \left( 500 \cdot n \right) \times \left( \frac{1}{500} \cdot n \right) \times \left( \log_2 n \right) \times \left( \log_{10} n \right) \times \left( \ln n \right)
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\[
\Theta(n^2) \quad \Theta(n^2 + n)
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Lower-order terms: The fastest-growing part of the computation dominates the total

\[ \Theta(n^2) \quad \Theta(n^2 + n) \quad \Theta(n^2 + 500 \cdot n + \log_2 n + 1000) \]
Comparing orders of growth (n is the problem size)
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$\Theta(b^n)$
Comparing orders of growth (n is the problem size)

\[ \Theta(b^n) \]  Exponential growth. Recursive \texttt{fib} takes
\[ \Theta(\phi^n) \] steps, where \[ \phi = \frac{1 + \sqrt{5}}{2} \approx 1.61828 \]
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Incrementing the problem scales R(n) by a factor
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Doubling the problem only increments \( R(n) \).
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$\Theta(1)$ Constant. The problem size doesn't matter
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