Announcements
Scheme Recursive Art Contest: Start Early!
Scheme Recursive Art Contest: Start Early!

Fall 2012 Featherweight Winner
176 Scheme Tokens
Scheme Recursive Art Contest: Start Early!

Fall 2012 Featherweight Winner
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Fall 2013 Heavyweight Winner
1857 Scheme Tokens
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Fall 2012 Featherweight Winner
176 Scheme Tokens

Fall 2013 Heavyweight Winner
1857 Scheme Tokens

Extra lecture on ray tracing
Wednesday 11/2 5:00pm
2060 VLSB
Dynamic Scope
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The way in which names are looked up in Scheme and Python is called lexical scope (or static scope) [You can see what names are in scope by inspecting the definition]
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**Lexical scope:** The parent of a frame is the environment in which a procedure was defined
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Special form to create dynamically scoped procedures (mu special form only exists in Project 4 Scheme)

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13
Tail Recursion
Functional Programming
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All functions are pure functions
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No re-assignment and no mutable data types
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But... no `for/while` statements! Can we make basic iteration efficient? Yes!
Recursion and Iteration in Python

In Python, recursive calls always create new active frames

\[ \text{factorial}(n, k) \text{ computes: } n! \times k \]
Recursion and Iteration in Python

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\[
\text{factorial}(n, k) \text{ computes: } n! * k
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def factorial(n, k):
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**factorial(n, k) computes**: $n! \times k$

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"Implementations of Scheme are required to be properly tail-recursive. This allows the execution of an iterative computation in constant space, even if the iterative computation is described by a syntactically recursive procedure."

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How? Eliminate the middleman!

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(Demo)
Tail Calls
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A procedure call that has not yet returned is **active**. Some procedure calls are **tail calls**. A Scheme interpreter should support an **unbounded number** of active tail calls using only a **constant** amount of space.
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Example: Length of a List
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A call expression is not a tail call if more computation is still required in the calling procedure.
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A call expression is not a tail call if more computation is still required in the calling procedure.

Linear recursive procedures can often be re-written to use tail calls.
Example: Length of a List

(define (length s)
  (if (null? s) 0
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\[
\text{(define (length s)}
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(define (length-tail s)
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(define (length-tail s)
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(define (length-iter s n)
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Example: Length of a List

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A call expression is not a tail call if more computation is still required in the calling procedure.

Linear recursive procedures can often be re-written to use tail calls.

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(define (length-tail s)
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  (length-iter s 0))
```
Example: Length of a List

\[
\text{(define \ (length \ s)}
\]
\[
\text{\ (if \ (null? \ s) \ 0 \ \text{Not a tail context})}
\]
\[
\text{\ (+ \ 1 \ (length \ (cdr \ s))) ) )}
\]

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\[
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\[
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\]
\[
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Eval with Tail Call Optimization
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(Demo)
Tail Recursion Examples
Which Procedures are Tail Recursive?

Which of the following procedures run in constant space? $\Theta(1)$

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;; Return whether s contains v.
(define (contains s v)
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;; Return whether s has any repeated elements.
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Map and Reduce
Example: Reduce
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(define (reduce procedure s start))
Example: Reduce

\[
\text{(define (reduce procedure s start)}
\]

\[
\text{(reduce \ast ' (3 4 5) 2)}
\]
Example: Reduce

(define (reduce procedure s start)

(reduce * '(3 4 5) 2) 120)
Example: Reduce

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\[
\text{(reduce (lambda (x y) (cons y x)) '(3 4 5) '(2))}
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Example: Reduce

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(reduce (lambda (x y) (cons y x)) '(3 4 5) '(2)) (5 4 3 2)
Example: Reduce

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\[(\text{(reduce procedure} \text{)}\]

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(define (reduce procedure s start)
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\begin{align*}
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\end{align*}
\]

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Example: Reduce

\[(\text{define } (\text{reduce } \text{procedure } s \text{ start}) )\]

\[
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\begin{array}{c}
(\text{reduce } \text{procedure } \\
(\text{cdr } s) \\
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\end{array}
) )
\]

\( (\text{reduce } \ast ' (3 4 5) 2) \quad 120 \)

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Recursive call is a tail call
Space depends on what procedure requires

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```
Example: Map with Only a Constant Number of Frames
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(define (map procedure s))
Example: Map with Only a Constant Number of Frames

```
(define (map procedure s)
  (if (null? s)
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Example: Map with Only a Constant Number of Frames

```
(define (map procedure s)
  (if (null? s)
      nil
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            (map procedure (cdr s))))
)

(map (lambda (x) (- 5 x)) (list 1 2))
```
Example: Map with Only a Constant Number of Frames

```
(define (map procedure s)
  (if (null? s)
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      (cons (procedure (car s))
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```

```
(define (map procedure s)
  (if (null? s)
      m
      (define (map-reverse s m)
        (if (null? s)
            m
            (define (map procedure s)
              (if (null? s)
                  nil
                  (cons (procedure (car s))
                        (map procedure (cdr s)))))))))
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        m
        (map-reverse (cdr s)
                     (map procedure s))))
)
```

Example: Map with Only a Constant Number of Frames

\[
\begin{align*}
\text{(define (map procedure \textit{s})} & \text{)} \\
& \text{(if (null? \textit{s}) nil)} \\
& \text{(cons (procedure (car \textit{s})) (map procedure (cdr \textit{s})))))
\end{align*}
\]

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\text{(map (lambda (x) (- 5 x)) (list 1 2))}
\]

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& \text{(if (null? \textit{s}) nil)} \\
& \text{(cons (procedure (car \textit{s})) (map procedure (cdr \textit{s}))))}
\end{align*}
\]

\[
\text{(define (map-reverse \textit{s} \textit{m})} & \text{)} \\
& \text{(if (null? \textit{s}) \textit{m})} \\
& \text{(map-reverse (cdr \textit{s}) (cons (procedure (car \textit{s}))}}
\]
Example: Map with Only a Constant Number of Frames

\[
\begin{align*}
\text{(define (map procedure s)} & \quad \text{\textcircled{constant} number of frames)} \\
\text{(if (null? s)} & \quad \text{\textcircled{constant} number of frames)} \\
& \quad \text{\textcircled{constant} number of frames)} \\
\text{\hspace{2em} nil} & \quad \text{\textcircled{constant} number of frames)} \\
\text{\hspace{2em} (cons (procedure (car s))))} & \quad \text{\textcircled{constant} number of frames)} \\
\text{\hspace{4em} (map procedure (cdr s)))} & \quad \text{\textcircled{constant} number of frames)} \\
\end{align*}
\]

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\text{\hspace{2em} nil} & \quad \text{\textcircled{constant} number of frames)} \\
\text{\hspace{2em} (cons (procedure (car s)))} & \quad \text{\textcircled{constant} number of frames)} \\
\text{\hspace{4em} (map procedure (cdr s)))} & \quad \text{\textcircled{constant} number of frames)} \\
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```
(define (map procedure s)
  (if (null? s)
      nil
      (cons (procedure (car s))
            (map procedure (cdr s)))))

(define (map-reverse s m)
  (if (null? s)
      m
      (map-reverse (cdr s)
                   (cons (procedure (car s))
                         m))
       )

(reverse (map-reverse s nil)))
```
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```

```
(define (reverse s)
)
```
Example: Map with Only a Constant Number of Frames

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\text{(define (map procedure s)} \ni\text{nil)} \\
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  (if (null? s)
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      (cons (procedure (car s))
            (map procedure (cdr s)))))

(map (lambda (x) (- 5 x)) (list 1 2))

(define (map-reverse s m)
  (if (null? s)
      m
      (map-reverse (cdr s)
                   (cons (procedure (car s))
                         m))
      ))

(reverse (map-reverse s nil)))

(define (reverse s)
  (define (reverse-iter s r)
    (if (null? s)
        r
        (reverse-iter (cdr s)
                      (cons (procedure (car s))
                            r))
      ))

(reverse s)
Example: Map with Only a Constant Number of Frames

(define (map procedure s)
  (if (null? s)
      nil
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(map (lambda (x) (- 5 x)) (list 1 2))
Example: Map with Only a Constant Number of Frames

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(define (map procedure s)
  (if (null? s)
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(define (reverse s)
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        r
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                            m))
    )
  )

(reverse (map-reverse s nil)))
```
Example: Map with Only a Constant Number of Frames

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(define (map procedure s)
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        (reverse-iter (cdr s)
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Example: Map with Only a Constant Number of Frames

\[
\text{(define (map procedure s)}
\begin{array}{l}
\text{(if (null? s)}
\text{nil)
\text{(cons (procedure (car s))}
\text{(map procedure (cdr s)))})
\end{array}
\text{)}
\]

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\text{(map (lambda (x) (- 5 x)) (list 1 2))}
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\text{(define (map-reverse s m)}
\begin{array}{l}
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\text{(map-reverse (cdr s))}
\text{(cons (procedure (car s))}
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\end{array}
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\text{(reverse (map-reverse s nil))}
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\text{(if (null? s)}
\text{r)
\text{(reverse-iter (cdr s))}
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\end{array}
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\text{(reverse-iter s nil))}
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Example: Map with Only a Constant Number of Frames

(define (map procedure s)
  (if (null? s)
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(map (lambda (x) (- 5 x)) (list 1 2))

(define (map-reverse s m)
  (if (null? s)
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\begin{align*}
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General Computing Machines
An Analogy: Programs Define Machines
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Programs specify the logic of a computational device
An Analogy: Programs Define Machines

Programs specify the logic of a computational device

factorial
An Analogy: Programs Define Machines

Programs specify the logic of a computational device

\[
\text{factorial} = \text{factorial} \times 1
\]

\[
\text{factorial} - 1 = 1
\]
An Analogy: Programs Define Machines

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Programs specify the logic of a computational device

\[ \text{factorial} \]

\[ 5 \rightarrow 4 = 1 \rightarrow 1 \rightarrow 120 \]

\[ 5 = \text{factorial}(5) = 5 \times 4 \times 3 \times 2 \times 1 = 120 \]
Interpreters are General Computing Machine
Interpreters are General Computing Machine

An interpreter can be parameterized to simulate any machine
Interpreters are General Computing Machine

An interpreter can be parameterized to simulate any machine

```
(define (factorial n)
  (if (zero? n) 1 (* n (factorial (- n 1)))))
```

5 → Scheme Interpreter → 120
Interpreters are General Computing Machine

An interpreter can be parameterized to simulate any machine

\[
5 \rightarrow \text{Scheme Interpreter} \rightarrow 120
\]

\[
\text{(define (factorial n)}
\text{ (if (zero? n) 1 (* n (factorial (- n 1))))})
\]

Our Scheme interpreter is a universal machine
Interpreters are General Computing Machine

An interpreter can be parameterized to simulate any machine.

```
(define (factorial n)
  (if (zero? n) 1 (* n (factorial (- n 1)))))
```

Our Scheme interpreter is a universal machine.

A bridge between the data objects that are manipulated by our programming language and the programming language itself.
Interpreters are General Computing Machine

An interpreter can be parameterized to simulate any machine

\[(\text{define } (\text{factorial } n) \ (if (\text{zero? } n) 1 (* n (\text{factorial } (- n 1))))\)

Our Scheme interpreter is a universal machine

A bridge between the data objects that are manipulated by our programming language and the programming language itself

Internally, it is just a set of evaluation rules