Announcements

- HW12 due Wednesday

- Scheme project, contest out
Review: Program Generator
A computer program is just a sequence of bits
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It is possible to enumerate all bit sequences
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```python
from itertools import product

def bitstrings():
    size = 0
    while True:
        tuples = product(('0', '1'), repeat=size)
        for elem in tuples:
            yield ''.join(elem)
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['', '0', '1', '00', '01', '10', '11', '000', '001', '010']
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Review: Function Streams
Given a stream of 1-argument functions, we can construct a function that is not in the stream, *assuming that all functions in the stream terminate*.
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Functions
Review: Function Streams

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def func(x):
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A mathematical function $f$ is *computable* if there exists a program (i.e. a Python function) `func` that computes it
Are all functions computable?
Computability

Are all functions computable?

More specifically, we hate infinite loops
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So if we have a program that computes the following function, we can run it on our programs to determine if they have infinite loops:
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\[
\text{haltsonallinputs} : \text{Programs} \rightarrow \{0, 1\},
\]

\[
\text{haltsonallinputs}(P) = \begin{cases} 
1 & \text{if } P \text{ halts on all inputs} \\
0 & \text{otherwise}
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Thus, we have to do something more clever, analyzing the program itself.
Turing
Let’s assume that we have a Python function `halts` that computes the mathematical function `halts`, written by someone more clever than us.
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Remember, we can pass a function itself as its argument. Thus, we can consider `halts(f, f)`; in other words, does function `f` halt when given itself as an argument? (This is just a thought experiment.)
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We can then define a new function, `turing`, which takes in 1 argument.
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\begin{verbatim}
def turing(f):
    if halts(f, f):
        while True:  # infinite loop
            pass
    else:
        return True  # halts
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`turing` will go into an infinite loop if \( f \) halts when given itself as an argument. Otherwise, `turing` returns `True`. 
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# * what?
Turing... what?

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print(turing(turing))  # * what?

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We have a contradiction! Our assumption that `halts` exists is false.
Bitstrings and Functions
Let’s develop another proof, assuming that we have a \texttt{halts} program that computes the mathematical function \textit{halts}
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Assume we have the following Python functions:

```python
def is_valid_python_function(bitstring):
    """Determine whether or not a bitstring represents a syntactically valid 1-argument Python function.""
```
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def is_valid_python_function(bitstring):
    """Determine whether or not a bitstring represents a syntactically valid 1-argument Python function.""

def bitstring_to_python_function(bitstring):
    """Coerce a bitstring representation of a Python function to the function itself."""
```
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def function_stream():
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def function_stream():
    """Return a stream of all valid 1-argument Python functions.""
    bitstring_stream = \texttt{iterator_to_stream(bitstrings())}
    valid_stream = \texttt{filter_stream(is_valid_python_function, bitstring_stream)}
```

\textit{On HW12}

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def function_stream():
    """Return a stream of all valid 1-argument Python functions.""
    bitstring_stream = iterator_to_stream(bitstrings())
    valid_stream = filter_stream(is_valid_python_function, bitstring_stream)
    return map_stream(bitstring_to_python_function, valid_stream)
```
Filtering Out Non-Terminating Programs
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Filtering Out Non-Terminating Programs

With `halts`, we can’t filter out programs that don’t halt on all input.

But we can filter out programs that don’t halt on a specific input.

Specifically, let’s make sure that a program halts on its index in the resulting stream of programs:

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def make_halt_checker():
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We now have a stream of programs that halt when given their own index as input.
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So halt_checker returns true on church, which means that church is in programs
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This results in an infinite loop, which means halt_checker will return false on church, since it does not halt given its own index
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So we made a false assumption somewhere
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\text{halts} : \text{Programs} \times \mathbb{N} \rightarrow \{0, 1\},
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\text{halts}(P, n) = \begin{cases} 
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We proved that $halts$ is uncomputable in Python, but our reasoning applies to all languages.

It is a fundamental limitation of all computers and programming languages.
Uncomputable Functions

It gets worse; not only can we not determine programatically whether or not a given program halts, we can’t determine *anything* “interesting” about the *behavior* of a program in general.
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Since we know we can’t write `halts`, our assumption that we can write `prints_something` is false.
Consequences
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For example, perfect anti-virus software is impossible.

- Anti-virus software must either miss some viruses (false negatives), mark some innocent programs as viruses (false positives), or fail to terminate on others.

We can’t write perfect security analyzers, optimizing compilers, etc.
Incompleteness Theorem
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- Given a finite set of axioms and inference rules, a program can check that each statement in a proof follows from the previous ones.

Thus, if a valid proof exists for a mathematical formula, then a computer can find it.
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If \( H(P, n) \) is provable or disprovable for all \( P \) and \( n \), then we can write a program to prove or disprove it by generating all proofs and checking each one to see if it proves or disproves \( H(P, n) \).
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Thus, there must be values of \(P\) and \(n\) for which \(H(P, n)\) is neither provable nor disprovable, or for which an incorrect result can be proven
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Thus, there must be values of \( P \) and \( n \) for which \( H(P, n) \) is neither provable nor disprovable, or for which an incorrect result can be proven

Thus, there are fundamental limitations not only to computation, but to mathematics itself!
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**os.system('python <file>')**: Directs the operating system to invoke a new instance of the Python interpreter.