

1 Recursion

A *recursive* function is a function that calls itself. Here's a recursive function:

Although we haven't finished defining `factorial`, we are still able to call it since the function body is not evaluated until the function is called. We do have one *base case*: when `n` is `0` or `1`. Now we can compute `factorial(2)` in terms of `factorial(1)`, and `factorial(3)` in terms of `factorial(2)`, and `factorial(4)` – well, you get the idea.

There are *three* common steps in a recursive definition:

1. *Figure out your base case*: What is the simplest argument we could possibly get? For example, `factorial(0)` is 1 by definition.
2. *Make a recursive call with a simpler argument*: Simplify your problem, and assume that a recursive call for this new problem will simply work. This is called the “leap of faith”. For `factorial`, we reduce the problem by calling `factorial(n-1)`.
3. *Use your recursive call to solve the full problem*: Remember that we are assuming the recursive call works. With the result of the recursive call, how can you solve the original problem you were asked? For `factorial`, we just multiply $(n - 1)!$ by n .

Questions

- 1.1 Create a recursive countdown function that takes in an integer `n` and prints out a countdown from `n` to 1. The function is defined on the next page.

First, think about a base case for the countdown function. What is the simplest input the problem could be given?

After you've thought of a base case, think about a recursive call with a smaller argument that approaches the base case. What happens if you call `countdown(n - 1)`?

```
def factorial(n):
    if n == 0 or n == 1:
        return 1
    else:
        return n * factorial(n-1)
```

Then, put the base case and the recursive call together, and think about where a print statement would be needed.

```
def countdown(n):
    """
    >>> countdown(3)
    3
    2
    1
    """
```

1.2 Is there an easy way to change `countdown` to count up instead?

2 Iteration vs. Recursion

We've written `factorial` recursively. Let's compare the two implementations:

```
def recursive_fact(n):
    if n == 0 or n == 1:
        return 1
    else:
        return n * recursive_fact(n-1)

def iterative_fact(n):
    total = 1
    while n > 1:
        total = total * n
        n = n - 1
    return total
```

Let's also compare `fibonacci`.

```
def recursive_fib(n):
    if n == 0:
        return 0
    elif n == 1:
        return 1
    else:
        return (recursive_fib(n-1)
                + recursive_fib(n-2))

def iterative_fib(n):
    prev, curr = 0, 1
    while n > 0:
        prev, curr = curr, prev + curr
        n = n - 1
    return prev
```

For the recursive version, we copied the definition of the Fibonacci sequence straight into code! The n th Fibonacci number is simply the sum of the two before it. In iteration, you need to keep track of more numbers and have a better understanding of the code.

Some code is easier to write iteratively and some recursively. Have fun experimenting with both!

Questions

- 2.1 Our Python interpreter is broken and `pow` no longer works! However, we can write our own replacement. Let's try writing it both iteratively and recursively. Both `expt_iter(base, power)` and `expt_rec(base, power)` implement the exponent function, so we should get the same answer regardless of which one we use. Assume that `power` is a non-negative integer.

```
def expt_iter(base, power):
    """ Implements the exponent function iteratively
    >>> expt_iter(2, 3)
    8
    >>> expt_iter(4, 0)
    1
    """
```

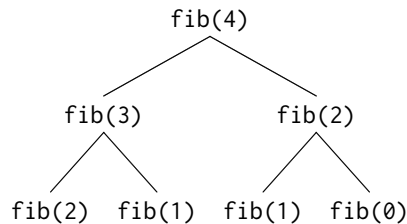
```
def expt_rec(base, power):
    """ Implements the exponent function recursively
    >>> expt_rec(3, 2)
    9
    """
```

3 Tree Recursion

Consider a function that requires more than one recursive call. A simple example is the previous recursive fibonacci function:

```
def fib(n):
    if n == 0:
        return 0
    elif n == 1:
        return 1
    else:
        return fib(n - 1) + fib(n - 2)
```

This type of recursion is called *tree recursion*, because it makes more than one recursive call in its recursive case. If we draw out the recursive calls, we see the recursive calls in the shape of an upside-down tree:



We could, in theory, use loops to write the same procedure. However, problems that are naturally solved using tree recursive procedures are generally difficult to write iteratively. As a general rule of thumb, whenever you need to try multiple possibilities at the same time, you should consider using tree recursion.

Questions

- 3.1 I want to go up a flight of stairs that has n steps. I can either take 1 or 2 steps each time. How many different ways can I go up this flight of stairs? Write a function `count_stair_ways` that solves this problem for me. Assume n is positive.

Before we start, what's the base case for this question? What is the simplest input?

What do `count_stair_ways(n - 1)` and `count_stair_ways(n - 2)` represent?

Use those two recursive calls to write the recursive case:

```
def count_stair_ways(n):
```

- 3.2 Consider an insect in an M by N grid. The insect starts at the bottom left corner, $(0, 0)$, and wants to end up at the top right corner $(M-1, N-1)$. The insect is only capable of moving right or up. Write a function `paths` that takes a grid length and width and returns the number of different paths the insect can take from the start to the goal. (There is a closed-form solution to this problem, but try to answer it procedurally using recursion.)

```
def paths(m, n):
    """
    >>> paths(2, 2)
    2
    >>> paths(117, 1)
    1
    """
```

- 3.3 The TAs want to print handouts for their students. However, for some unfathomable reason, both the printers are broken; the first printer only prints multiples of n_1 , and the second printer only prints multiples of n_2 . Help the TAs figure out whether or not it is possible to print an exact number of handouts!

First try to solve without a helper function. Also try to solve using a helper function and adding up to the sum.

```
def has_sum(total, n1, n2):
    """
    >>> has_sum(1, 3, 5)
    False
    >>> has_sum(5, 3, 5) # 0(3) + 1(5) = 5
    True
    >>> has_sum(11, 3, 5) # 2(3) + 1(5) = 11
    True
    """
```

- 3.4 The next day, the printers break down even more! Each time they are used, Printer A prints a random x copies $50 \leq x \leq 60$, and Printer B prints a random y copies $130 \leq y \leq 140$. The TAs also relax their expectations: they are satisfied as long as they get at least `lower`, but no more than `upper`, copies printed. (More than `upper` copies is unacceptable because it wastes too much paper.)

Hint: Try using a helper function.

```
def sum_range(lower, upper):
    """
    >>> sum_range(45, 60) # Printer A prints within this range
    True
    >>> sum_range(40, 55) # Printer A can print a number 56-60
    False
    >>> sum_range(170, 201) # Printer A + Printer B will print
    ... # somewhere between 180 and 200 copies total
    True
    """
```