CS61B Lecture #31

Today:

- Pseudo-random Numbers (Chapter 11)
- What use are random sequences?
- What *are* "random sequences"?
- Pseudo-random sequences.
- How to get one.
- Relevant Java library classes and methods.
- Random permutations.

Why Random Sequences?

- Choose statistical samples
- Simulations
- Random algorithms
- Cryptography:
  - Choosing random keys
  - Generating streams of random bits (e.g., SSL xor's your data with a regeneratable, pseudo-random bit stream that only you and the recipient can generate).
- And, of course, games

What Is a "Random Sequence"?

- How about: "a sequence where all numbers occur with equal frequency"?
  - Like 1, 2, 3, 4, ...?
- Well then, how about: "an unpredictable sequence where all numbers occur with equal frequency?"
  - Like 0, 0, 0, 1, 1, 2, 2, 2, 2, 2, 3, 4, 4, 0, 1, 1, 1,...?
- Besides, what is wrong with 0, 0, 0, 0, ... anyway? Can't that occur by random selection?

Pseudo-Random Sequences

- Even if definable, a "truly" random sequence is difficult for a computer (or human) to produce.
- For most purposes, need only a sequence that satisfies certain statistical properties, even if deterministic.
- Sometimes (e.g., cryptography) need sequence that is *hard* or *impractical* to predict.
- *Pseudo-random sequence:* deterministic sequence that passes some given set of statistical tests.
- For example, look at lengths of *runs:* increasing or decreasing contiguous subsequences.
- Unfortunately, statistical criteria to be used are quite involved. For details, see Knuth.

Generating Pseudo-Random Sequences

- Not as easy as you might think.
- Seemingly complex jumbling methods can give rise to bad sequences.
- *Linear congruential method* is a simple method that has withstood test of time:

$$
\begin{aligned}
X_0 &= \textit{arbitrary seed} \\
X_i &= (aX_{i-1} + c) \bmod m, \quad i > 0
\end{aligned}
$$

- Usually, m is large power of 2.
- For best results, want $a \equiv 5 \bmod 8$, and a, c, m with no common factors.
- This gives generator with a *period of* m (length of sequence before repetition), and reasonable *potency* (measures certain dependencies among adjacent X_i.)
- Also want bits of a to "have no obvious pattern" and pass certain other tests (see Knuth).
- Java uses $a = 25214903917$, $c = 11$, $m = 2^{48}$, to compute 48-bit pseudo-random numbers but I haven't checked to see how good this is.

What Can Go Wrong?

- Short periods, many impossible values: E.g., a, c, m even.
- Obvious patterns. E.g., just using lower 3 bits of X_i in Java's 48-bit generator, to get integers in range 0 to 7. By properties of modular arithmetic,

$$
\begin{aligned}
X_i \bmod 8 &= (25214903917X_{i-1} + 11 \bmod 2^{48}) \bmod 8 \\
&= (5(X_{i-1} \bmod 8) + 3) \bmod 8
\end{aligned}
$$

  so we have a period of 8 on this generator; sequences like

$$0, 1, 3, 7, 1, 2, 7, 1, 4, \ldots$$

  are impossible. This is why Java doesn't give you the raw 48 bits.
- Bad potency leads to bad correlations.
  - E.g. Take $c = 0$, $a = 65539$, $m = 2^{31}$, and make 3D points: $(X_i/S, X_{i+1}/S, X_{i+2}/S)$, where S scales to a unit cube.
  - Points will be arranged in parallel planes with voids between.
  - So, "random points" won't ever get near many points in the cube.

Other Generators

- Additive generator:

$$
X_n = \begin{cases} \textit{arbitary value}, & n < 55 \\ (X_{n-24} + X_{n-55}) \bmod 2^e, & n \geq 55 \end{cases}
$$

- Other choices than 24 and 55 possible.
- This one has period of $2^f(2^{55} - 1)$, for some $f < e$.
- Simple implementation with circular buffer:

```
i = (i+1) % 55;
X[i] += X[(i+31) % 55];  // Why +31 (55-24) instead of -24?
return X[i];   /* modulo 2^32 */
```

- where `X[0 .. 54]` is initialized to some "random" initial seed values.

Adjusting Range and Distribution

- Given raw sequence of numbers, X_i, from above methods in range (e.g.) 0 to 2^{48}, how to get uniform random integers in range 0 to $n - 1$?
- If $n = 2^k$, is easy: use top k bits of next X_i (bottom k bits not as "random")
- For other n, be careful of slight biases at the ends. For example, if we compute $X_i/(2^{48}/n)$ using all integer division, and if $(2^{48}/n)$ doesn't come out even, then you can get n as a result (which you don't want).
- Easy enough to fix with floating point, but can also do with integers; one method (used by Java for type `int`):

```
/** Random integer in the range 0 .. n-1, n>0. */
int nextInt (int n) {
  long X = next random long (0 ≤ X < 2^48);
  if (n is 2^k for some k) return top k bits of X;
  int MAX = largest multiple of n that is < 2^48;
  while (X_i >= MAX)  X = next random long (0 ≤ X < 2^48);
  return X_i / (MAX/n);
}
```

Arbitrary Bounds

- How to get arbitrary range of integers (L to U)?
- To get random `float`, x in range $0 \le x < d$, compute

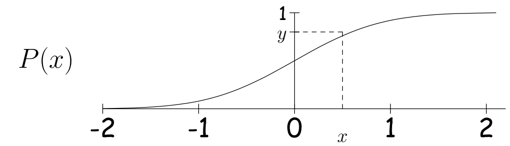
```
return d*nextInt (1<<24) / (1<<24);
```

- Random `double` a bit more complicated: need two integers to get enough bits.

```
long bigRand = ((long) nextInt(1<<26) << 27) + (long) nextInt(1<<27);
return d * bigRand / (1L << 53);
```


Other Distributions

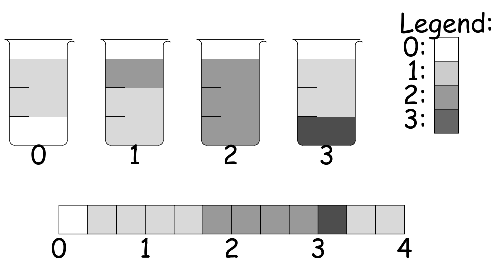
- Can also turn uniform random integers into arbitrary other distributions, like the Gaussian.



- Curve is the desired probability distribution ($P(x)$ is the probability that a certain random variable is $\le x$.)
- Choose y uniformly between 0 and 1, and the corresponding x will be distributed according to P.

Computing Arbitrary Discrete Distribution

- Example from book: want integer values X_i with $\Pr(X_i = 0) = 1/12$, $\Pr(X_i = 1) = 1/2$, $\Pr(X_i = 2) = 1/3$, $\Pr(X_i = 3) = 1/12$:



- To get desired probabilities, choose floating-point number, $0 \le R_i < 4$, and see what color you land on.
- ≤ 2 colors in each beaker $\equiv \le 2$ colors between i and $i+1$.

```
return (R_i % 1.0 > v[(int) R_i])
          ? top[(int) R_i]
          : bot[R_i];
```

where

```
v = { 1.0/3.0, 2.0/3.0, 0, 1.0/3.0 };
top = { 1, 2, 2, 1 },
bot = { 0, 1, /* ANY */ 0, 3 };
```


Java Classes

- `Math.random()`: random double in $[0..1)$.
- Class `java.util.Random`: a random number generator with constructors:

  Random() generator with "random" seed (based on time).

  Random(seed) generator with given starting value (reproducible).
- Methods

  next(k) k-bit random integer

  nextInt(n) `int` in range $[0..n)$.

  nextLong() random 64-bit integer.

  nextBoolean(), nextFloat(), nextDouble() Next random values of other primitive types.

  nextGaussian() normal distribution with mean 0 and standard deviation 1 ("bell curve").
- `Collections.shuffle(`L, R`)` for list R and `Random` R permutes L randomly (using R).

Shuffling

- A *shuffle* is a random permutation of some sequence.
- Obvious dumb technique for sorting N-element list:
  - Generate N random numbers
  - Attach each to one of the list elements
  - Sort the list using random numbers as keys.
- Can do quite a bit better:

```
void shuffle (List L, Random R) {
   for (int i = L.size (); i > 0; i -= 1)
       swap element i-1 of L with element R.nextInt (i) of L;
}
```

- Example:

Swap items	0	1	2	3	4	5
Start	A♣	2♣	3♣	A♡	2♡	3♡
$5 \Longleftrightarrow 1$	A♣	3♡	3♣	A♡	2♡	2♣
$4 \Longleftrightarrow 2$	A♣	3♡	2♡	A♡	3♣	2♣

Swap items	0	1	2	3	4	5
$3 \Longleftrightarrow 3$	A♣	3♡	2♡	A♡	3♣	2♣
$2 \Longleftrightarrow 0$	2♡	3♡	A♣	A♡	3♣	2♣
$1 \Longleftrightarrow 0$	3♡	2♡	A♣	A♡	3♣	2♣

Random Selection

- Same technique would allow us to select N items from list:

```
/** Permute L and return sublist of K>=0 randomly
 *  chosen elements of L, using R as random source. */
List select (List L, int k, Random R) {
  for (int i = L.size (); i+k > L.size (); i -= 1)
    swap element i-1 of L with element
      R.nextInt (i) of L;
  return L.sublist (L.size ()-k, L.size ());
}
```

- Not terribly efficient for selecting random sequence of K distinct integers from $[0..N)$, with $K \ll N$.

Alternative Selection Algorithm (Floyd)

```
/** Random sequence of K distinct integers
 *  from 0..N-1, 0<=K<=N. */
IntList selectInts(int N, int K, Random R)
{
  IntList S = new IntList();

  for (int i = N-K; i < N; i += 1) {
    // All values in S are < i
    int s = R.randInt(i+1); // 0 <= s <= i < N
    if (s == S.get(j) for some j)
      // Insert value i (which can't be there
      // yet) after the s (i.e., at a random
      // place other than the front)
      S.add (j+1, i);
    else
      // Insert random value s at front
      S.add (0, s);
  }
  return S;
}
```

Example

i	s	S
5	4	[4]
6	2	[2, 4]
7	5	[5, 2, 4]
8	5	[5, 8, 2, 4]
9	4	[5, 8, 2, 4, 9]

`selectRandomIntegers (10, 5, R)`