CS61B Lectures #27–28

Today:

- Sorting algorithms: why?

- Insertion, Shell's, Heap, Merge sorts

- Quicksort

- Selection

- Distribution counting, radix sorts

Readings: Today: *DS(IJ), Chapter 8;* Next topic: Chapter 9.

Purposes of Sorting

- Sorting supports searching

- Binary search standard example

- Also supports other kinds of search:

  - Are there two equal items in this set?
  - Are there two items in this set that both have the same value for property X?
  - What are my nearest neighbors?

- Used in numerous unexpected algorithms, such as convex hull (smallest convex polygon enclosing set of points).

Some Definitions

- A sort is a *permutation* (re-arrangement) of a sequence of elements that brings them into order, according to some *total order*. A total order, $\preceq$, is:

  - **Total:** $x \preceq y$ or $y \preceq x$ for all x, y.
  - **Reflexive:** $x \preceq x$;
  - **Antisymmetric:** $x \preceq y$ and $y \preceq x$ iff $x = y$.
  - **Transitive:** $x \preceq y$ and $y \preceq z$ implies $x \preceq z$.

- However, our orderings may allow unequal items to be equivalent:

  - E.g., can be two dictionary definitions for the same word: if entries sorted only by word, then sorting could put either entry first.
  - A sort that does not change the relative order of equivalent entries is called *stable*.

Classifications

- *Internal sorts* keep all data in primary memory

- *External sorts* process large amounts of data in batches, keeping what won't fit in secondary storage (in the old days, tapes).

- *Comparison-based* sorting assumes only thing we know about keys is order

- *Radix sorting* uses more information about key structure.

- *Insertion sorting* works by repeatedly inserting items at their appropriate positions in the sorted sequence being constructed.

- *Selection sorting* works by repeatedly selecting the next larger (smaller) item in order and adding it one end of the sorted sequence being constructed.

Sorting by Insertion

- Simple idea:

  - starting with empty sequence of outputs.
  - add each item from input, *inserting* into output sequence at right point.

- Very simple, good for small sets of data.

- With vector or linked list, time for find + insert of one item is at worst $\Theta(k)$, where k is # of outputs so far.

- So gives us $O(N^2)$ algorithm. Can we say more?

Inversions

- Can run in $\Theta(N)$ comparisons if already sorted.

- Consider a typical implementation for arrays:

```
for (int i = 1; i < A.length; i += 1) {
   int j;
   Object x = A[i];
   for (j = i-1; j >= 0; j -= 1) {
     if (A[j].compareTo (x) <= 0)   /* (1) */
       break;
     A[j+1] = A[j];
   }
   A[j+1] = x;
}
```
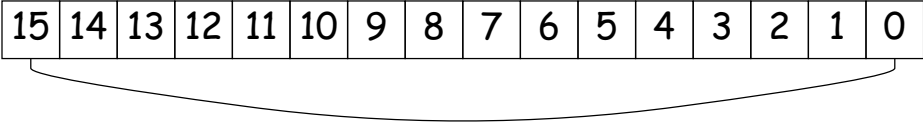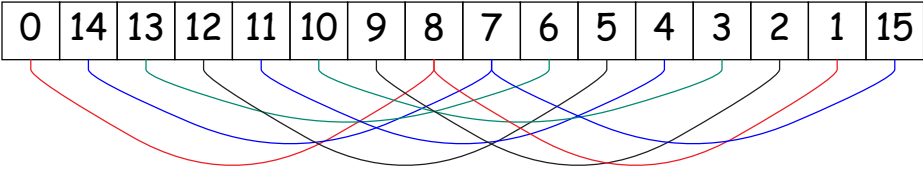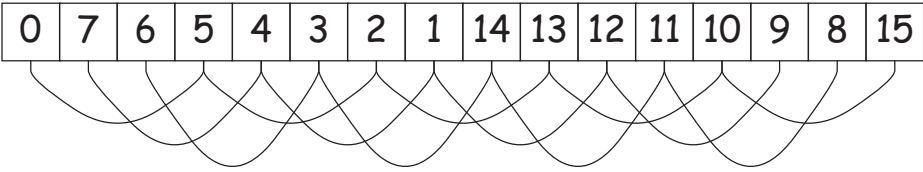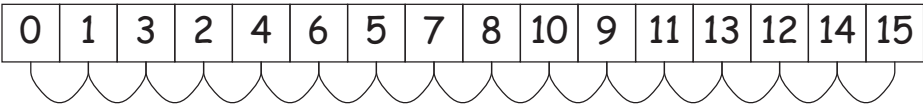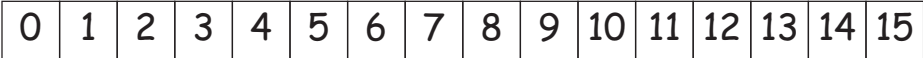
- #times (1) executes $\approx$ how far x must move.

- If all items within K of proper places, then takes $O(KN)$ operations.

- Thus good for any amount of *nearly sorted* data.

- One measure of unsortedness: # of *inversions*: pairs that are out of order (= 0 when sorted, $N(N-1)/2$ when reversed).

- Each step of j decreases inversions by 1.

Shell's sort

Idea: Improve insertion sort by first sorting *distant* elements:

- First sort subsequences of elements $2^k - 1$ apart:
    - sort items #$0,\ 2^k - 1,\ 2(2^k - 1),\ 3(2^k - 1),\ \ldots$, then
    - sort items #$1,\ 1 + 2^k - 1,\ 1 + 2(2^k - 1),\ 1 + 3(2^k - 1),\ \ldots$, then
    - sort items #$2,\ 2 + 2^k - 1,\ 2 + 2(2^k - 1),\ 2 + 3(2^k - 1),\ \ldots$, then
    - etc.
    - sort items #$2^k - 2,\ 2(2^k - 1) - 1,\ 3(2^k - 1) - 1,\ \ldots$,
    - Each time an item moves, can reduce #inversions by as much as $2^k + 1$.

- Now sort subsequences of elements $2^{k-1} - 1$ apart:
    - sort items #$0,\ 2^{k-1} - 1,\ 2(2^{k-1} - 1),\ 3(2^{k-1} - 1),\ \ldots$, then
    - sort items #$1,\ 1 + 2^{k-1} - 1,\ 1 + 2(2^{k-1} - 1),\ 1 + 3(2^{k-1} - 1),\ \ldots$,
    - ⋮

- End at plain insertion sort ($2^0 = 1$ apart), but with most inversions gone.

- Sort is $\Theta(N^{1.5})$ (take CS170 for why!).

Example of Shell's Sort

	#I	#C
15 14 13 12 11 10 9 8 7 6 5 4 3 2 1 0	120	1
0 14 13 12 11 10 9 8 7 6 5 4 3 2 1 15	91	10
0 7 6 5 4 3 2 1 14 13 12 11 10 9 8 15	42	20
0 1 3 2 4 6 5 7 8 10 9 11 13 12 14 15	4	19
0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15	0	-

I: Inversions left.
C: Comparisons needed to sort subsequences.

Sorting by Selection: Heapsort

Idea: Keep selecting smallest (or largest) element.

- Really bad idea on a simple list or vector.

- But we've already seen it in action: use heap.

- Gives $O(N \lg N)$ algorithm (N remove-first operations).

- Since we remove items from end of heap, we can use that area to accumulate result:

original:	19	0	-1	7	23	2	42
heapified:	42	23	19	7	0	2	-1
	23	7	19	-1	0	2	42
	19	7	2	-1	0	23	42
	7	0	2	-1	19	23	42
	2	0	-1	7	19	23	42
	0	-1	2	7	19	23	42
	-1	0	2	7	19	23	42

Merge Sorting

Idea: Divide data in 2 equal parts; recursively sort halves; merge results.

- Already seen analysis: $\Theta(N \lg N)$.

- Good for *external sorting:*

  - First break data into small enough chunks to fit in memory and sort.

  - Then repeatedly merge into bigger and bigger sequences.

  - Can merge K sequences of *arbitrary size* on secondary storage using $\Theta(K)$ storage.

- For internal sorting, can use *binomial comb* to orchestrate:

Illustration of Internal Merge Sort

L: (9, 15, 5, 3, 0, 6, 10, -1, 2, 20, 8)

```
0: 0
1: 0
2: 0
3: 0
```
0 elements processed

```
0: 1 •──→ (9)
1: 0
2: 0
3: 0
```
1 element processed

```
0: 0
1: 1 •──→ (9, 15)
2: 0
3: 0
```
2 elements processed

```
0: 1 •──→ (5)
1: 1 •──→ (9, 15)
2: 0
3: 0
```
3 elements processed

```
0: 0
1: 0
2: 1 •──→ (3, 5, 9, 15)
3: 0
```
4 elements processed

```
0: 0
1: 1 •──→ (0, 6)
2: 1 •──→ (3, 5, 9, 15)
3: 0
```
6 elements processed

```
0: 1 •──→ (8)
1: 1 •──→ (2, 20)
2: 0
3: 1 •──→ (-1, 0, 3, 5, 6, 9, 10, 15)
```
11 elements processed

Quicksort: Speed through Probability

Idea:

- *Partition* data into pieces: everything $>$ a *pivot* value at the high end of the sequence to be sorted, and everything $\leq$ on the low end.

- Repeat recursively on the high and low pieces.

- For speed, stop when pieces are "small enough" and do insertion sort on the whole thing.

- Reason: insertion sort has low constant factors. By design, no item will move out of its will move out of its piece [why?], so when pieces are small, #inversions is, too.

- Have to choose pivot well. E.g.: *median* of first, last and middle items of sequence.

Example of Quicksort

- In this example, we continue until pieces are size ≤ 4.

- Pivots for next step are starred. Arrange to move pivot to dividing line each time.

- Last step is insertion sort.

16	10	13	18	-4	-7	12	-5	19	15	0	22	29	34	-1*

-4	-5	-7		-1		18	13	12	10	19	15	0	22	29	34	16*

-4	-5	-7		-1		15	13	12*	10	0		16		19*	22	29	34	18

-4	-5	-7		-1		10	0		12		15	13		16		18		19		29	34	22

- Now everything is "close to" right, so just do insertion sort:

-7	-5	-4	-1	0	10	12	13	15	16	18	19	22	29	34

Performance of Quicksort

- Probabalistic time:

  - If choice of pivots good, divide data in two each time: $\Theta(N \lg N)$ with a good constant factor relative to merge or heap sort.

  - If choice of pivots bad, most items on one side each time: $\Theta(N^2)$.

  - $\Omega(N \lg N)$ in best case, so insertion sort better for nearly ordered input sets.

- Interesting point: randomly shuffling the data before sorting makes $\Omega(N^2)$ time *very* unlikely!

Quick Selection

The Selection Problem: for given k, find k^{th} smallest element in data.

- Obvious method: sort, select element #k, time $\Theta(N \lg N)$.

- If $k \leq$ some constant, can easily do in $\Theta(N)$ time:

    - Go through array, keep smallest k items.

- *Get probably $\Theta(N)$ time* for all k by adapting quicksort:

    - Partition around some pivot, p, as in quicksort, arrange that pivot ends up at dividing line.

    - Suppose that in the result, pivot is at index m, all elements $\leq$ pivot have indicies $\leq m$.

    - If $m = k$, you're done: p is answer.

    - If $m > k$, recursively select k^{th} from left half of sequence.

    - If $m < k$, recursively select $(k - m - 1)^{\text{th}}$ from right half of sequence.

Selection Example

Problem: Find just item #10 in the sorted version of array:

Initial contents:

51	60	21	-4	37	4	49	10	40*	59	0	13	2	39	11	46	31

  0

Looking for #10 to left of pivot 40:

13	31	21	-4	37	4*	11	10	39	2	0		40		59	51	49	46	60

  0

Looking for #6 to right of pivot 4:

-4	0	2		4		37	13	11	10	39	21	31*		40		59	51	49	46	60

      4

Looking for #1 to right of pivot 31:

-4	0	2		4		21	13	11	10		31		39	37		40		59	51	49	46	60

                          9

Just two elements; just sort and return #1:

-4	0	2		4		21	13	11	10		31		37	39		40		59	51	49	46	60

                          9

Result: 39

Selection Performance

- For this algorithm, if m roughly in middle each time, cost is

$$C(N) = \begin{cases} 1, & \text{if } N = 1, \\ N + C(N/2), & \text{otherwise.} \end{cases}$$
$$= N + N/2 + \ldots + 1$$
$$= 2N - 1 \in \Theta(N)$$

- But in worst case, get $\Theta(N^2)$, as for quicksort.

- By another, non-obvious algorithm, can get $\Theta(N)$ worst-case time for all k (take CS170).

Better than N lg N?

- Can prove that *if all you can do to keys is compare them* then sorting must take $\Omega(N \lg N)$.

- Basic idea: there are $N!$ possible ways the input data could be scrambled.

- Therefore, your program must be prepared to do $N!$ different combinations of move operations.

- Therefore, there must be $N!$ possible combinations of outcomes of all the **if** tests in your program (we're assuming that comparisons are 2-way).

- Since each **if** test goes two ways, number of possible different outcomes for k **if** tests is 2^k.

- Thus, need enough tests so that $2^k > N!$, which means $k \in \Omega(\lg N!)$.

- Using Stirling's approximation,

$$m! \in \sqrt{2\pi m} \left(\frac{m}{e}\right)^m \left(1 + \Theta\left(\frac{1}{m}\right)\right),$$

this tells us that

$$k \in \Omega(N \lg N).$$

Beyond Comparison: Distribution Counting

- But suppose can do more than compare keys?

- For example, how can we sort a set of N integer keys whose values range from 0 to kN, for some small constant k?

- One technique: *count* the number of items < 1, < 2, etc.

- If $M_p =$ #items with value $< p$, then in sorted order, the jth item with value p must be #$M_p + j$.

- Gives *linear-time* algorithm.

Distribution Counting Example

- Suppose all items are between 0 and 9 as in this example:

7	0	4	0	9	1	9	1	9	5	3	7	3	1	6	7	4	2	0

3	3	1	2	2	1	1	3	0	3	*Counts*
0	1	2	3	4	5	6	7	8	9	

0	3	6	7	9	11	12	13	16	16	*Running sum*
<0	<1	<2	<3	<4	<5	<6	<7	<8	<9	

0	0	0	1	1	1	2	3	3	4	4	5	6	7	7	7	9	9	9
0			3			6			9		11	12	13			16		

- "Counts" line gives # occurrences of each key.

- "Running sum" gives cumulative count of keys $\leq$ each value...

- ...which tells us where to put each key:

- The first instance of key k goes into slot m, where m is the number of key instances that are $< k$.

Radix Sort

Idea: Sort keys *one character at a time.*

- Can use distribution counting for each digit.

- Can work either right to left (LSD radix sort) or left to right (MSD radix sort)

- LSD radix sort is venerable: used for punched cards.

Initial: set, cat, cad, con, bat, can, be, let, bet

```
                                          bet
                                          let                                       bat   bet
  Pass 1                                  bat                      Pass 2           cat   let
                              can   cat                                             can   set
(by char #2)      be    cad   con   set                          (by char #1)      cad   be    con
                  ‾‾‾   ‾‾‾   ‾‾‾   ‾‾‾                                            ‾‾‾   ‾‾‾   ‾‾‾
                  '␣'   'd'   'n'   't'                                             'a'   'e'   'o'
         be, cad, con, can, set, cat, bat, let, bet                      cad, can, cat, bat, be, set, let, bet, con
```

```
                                          con
                              bet   cat
  Pass 3                      be    can
(by char #0)                  bat   cad   let   set
                              ‾‾‾   ‾‾‾   ‾‾‾   ‾‾‾
                              'b'   'c'   'l'   's'
                  bat, be, bet, cad, can, cat, con, let, set
```

MSD Radix Sort

- A bit more complicated: must keep lists from each step separate

- But, can stop processing 1-element lists

A	posn
$\star$ set, cat, cad, con, bat, can, be, let, bet	0
$\star$ bat, be, bet / cat, cad, con, can / let / set	1
bat / $\star$ be, bet / cat, cad, con, can / let / set	2
bat / be / bet / $\star$ cat, cad, con, can / let / set	1
bat / be / bet / $\star$ cat, cad, can / con / let / set	2
bat / be / bet / cad / can / cat / con / let / set	

Performance of Radix Sort

- Radix sort takes $\Theta(B)$ time where B is *total size of the key data*.

- Have measured other sorts as function of #records.

- How to compare?

- To have N different records, must have keys at least $\Theta(\lg N)$ long [why?]

- Furthermore, comparison actually takes time $\Theta(K)$ where K is size of key in worst case [why?]

- So $N \lg N$ comparisons really means $N(\lg N)^2$ operations.

- While radix sort takes $B = N \lg N$ time.

- On the other hand, must work to get good constant factors with radix sort.

And Don't Forget Search Trees

Idea: A search tree is in sorted order, when read in inorder.

- Need *balance* to really use for sorting [next topic].

- Given balance, same performance as heapsort: N insertions in time $\lg N$ each, plus $\Theta(N)$ to traverse, gives

$$\Theta(N + N \lg N) = \Theta(N \lg N)$$

Summary

- Insertion sort: $\Theta(Nk)$ comparisons and moves, where k is maximum amount data is displaced from final position.

  – Good for small datasets or almost ordered data sets.

- Quicksort: $\Theta(N \lg N)$ with good constant factor if data is not pathological. Worst case $O(N^2)$.

- Merge sort: $\Theta(N \lg N)$ guaranteed. Good for external sorting.

- Heapsort, treesort with guaranteed balance: $\Theta(N \lg N)$ guaranteed.

- Radix sort, distribution sort: $\Theta(B)$ (number of bytes). Also good for external sorting.