What Are the Questions?

• Cost is a principal concern throughout engineering:
  
  “An engineer is someone who can do for a dime what any fool can do for a dollar.”

• Cost can mean
  
  - \textit{Operational cost} (for programs, time to run, space requirements).
  - \textit{Development costs}: How much engineering time? When delivered?
  - \textit{Maintenance costs}: Upgrades, bug fixes.
  - \textit{Costs of failure}: How robust? How safe?

• Is this program \textit{fast enough}? Depends on:
  
  - \textit{For what purpose};
  - \textit{What input data}.

• How much space (memory, disk space)?
  
  - Again depends on what input data.

• How will it \textit{scale}, as input gets big?
Enlightening Example

Problem: Scan a text corpus (say $10^7$ bytes or so), and find and print the 20 most frequently used words, together with counts of how often they occur.

• Solution 1 (Knuth): Heavy-Duty data structures
  - Hash Trie implementation, randomized placement, pointers galore, several pages long.

• Solution 2 (Doug McIlroy): UNIX shell script:

  tr -c -s '[:alpha:]' '[\n*]' < FILE | 
  sort | 
  uniq -c | 
  sort -n -r -k 1,1 | 
  sed 20q

• Which is better?
  - #1 is much faster,
  - but #2 took 5 minutes to write and processes 30MB in < 6 sec.
  - I pick #2.

• In very many cases, almost anything will do: Keep It Simple.
Cost Measures (Time)

• **Wall-clock or execution** time
  - You can do this at home:
    
    ```
    time java FindPrimes 1000
    ```
  - Advantages: easy to measure, meaning is obvious.
  - Appropriate where time is critical (real-time systems, e.g.).
  - Disadvantages: applies only to specific data set, compiler, machine, etc.

• **Dynamic statement counts** of # of times statements are executed:
  - Advantages: more general (not sensitive to speed of machine).
  - Disadvantages: doesn’t tell you actual time, still applies only to specific data sets.

• **Symbolic execution times:**
  - That is, *formulas* for execution times as functions of input size.
  - Advantages: applies to all inputs, makes scaling clear.
  - Disadvantage: practical formula must be approximate, may tell very little about actual time.
Asymptotic Cost

- Symbolic execution time lets us see *shape* of the cost function.
- Since we are approximating anyway, pointless to be precise about certain things:
  - *Behavior on small inputs*:
    * Can always pre-calculate some results.
    * Times for small inputs not usually important.
    * Often more interested in *asymptotic behavior* as input size becomes very large.
  - *Constant factors* (as in “off by factor of 2”):
    * Just changing machines causes constant-factor change.
- How to abstract away from (i.e., ignore) these things?
Handy Tool: Order Notation

- Idea: Don’t try to produce specific functions that specify size, but rather families of functions with similar magnitudes.
- Then say something like “$f$ is bounded by $g$ if it is in $g$’s family.”
- For any function $g(x)$, the functions $2g(x), 0.5g(x)$, or for any $K > 0, K \cdot g(x)$, all have the same “shape”. So put all of them into $g$’s family.
- Any function $h(x)$ such that $h(x) = K \cdot g(x)$ for $x > M$ (for some constant $M$) has $g$’s shape “except for small values.” So put all of these in $g$’s family.
- For upper limits, throw in all functions whose absolute value is everywhere $\leq$ some member of $g$’s family. Call this set $O(g)$ or $O(g(n))$.
- Or, for lower limits, throw in all functions whose absolute values is everywhere $\geq$ some member of $g$’s family. Call this set $\Omega(g)$.
- Finally, define $\Theta(g) = O(g) \cap \Omega(g)$—the set of functions bracketed in magnitude by two members of $g$’s family.
Big Oh

• **Goal:** Specify bounding from above.

\[ f(x) \leq 2g(x) \text{ as long as } x > 1, \]

• So \( f(x) \) is in \( g \)'s “bounded-above family,” written

\[ f(x) \in O(g(x)), \]

• ... even though (in this case) \( f(x) > g(x) \) everywhere.
Big Omega

- **Goal**: Specify bounding from below:

  \[ M = 1 \]

  \[ g(x) \]

  \[ f'(x) \]

  \[ 0.5g(x) \]

  - Here, \( f'(x) \geq \frac{1}{2}g(x) \) as long as \( x > 1 \),
  - So \( f'(x) \) is in \( g \)'s “bounded-below family,” written
    \[ f'(x) \in \Omega(g(x)) \],
  - \ldots even though \( f(x) < g(x) \) everywhere.
Big Theta

• In the two previous slides, we not only have \( f(x) \in O(g(x)) \) and \( f'(x) \in \Omega(g(x)) \), ...

• ...but also \( f(x) \in \Omega(g(x)) \) and \( f'(x) \in O(g(x)) \).

• We can summarize this all by saying \( f(x) \in \Theta(g(x)) \) and \( f'(x) \in \Theta(g(x)) \).
Aside: Various Mathematical Pedantry

- Technically, if I am going to talk about \( O(\cdot) \), \( \Omega(\cdot) \) and \( \Theta(\cdot) \) as sets of functions, I really should write, for example,

\[
f \in O(g) \quad \text{instead of} \quad f(x) \in O(g(x))
\]

- In effect, \( f(x) \in O(g(x)) \) is short for \( \lambda x. f(x) \in O(\lambda x. g(x)) \).

- The standard notation outside this course, in fact, is \( f(x) = O(g(x)) \), but personally, I think that’s a serious abuse of notation.
How We Use Order Notation

• Elsewhere in mathematics, you’ll see $O(\ldots)$, etc., used generally to specify bounds on functions.

• For example,

$$\pi(N) = \Theta\left(\frac{N}{\ln N}\right)$$

which I would prefer to write

$$\pi(N) \in \Theta\left(\frac{N}{\ln N}\right)$$

(Here, $\pi(N)$ is the number of primes less than or equal to $N$.)

• Also, you’ll see things like

$$f(x) = x^3 + x^2 + O(x) \quad \text{(or } f(x) \in x^4 + x^2 + O(x)),$$

meaning that $f(x) = x^3 + x^2 + g(x)$ where $g(x) \in O(x)$.

• For our purposes, the functions we will be bounding will be cost functions: functions that measure the amount of execution time or the amount of space required by a program or algorithm.
Why It Matters

- *Computer scientists often talk as if constant factors didn't matter at all, only the difference of $\Theta(N)$ vs. $\Theta(N^2)$.*

- *In reality they do matter, but at some point, constants always get swamped.*

<table>
<thead>
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<th>$n$</th>
<th>$16 \lg n$</th>
<th>$\sqrt{n}$</th>
<th>$n$</th>
<th>$n \lg n$</th>
<th>$n^2$</th>
<th>$n^3$</th>
<th>$2^n$</th>
</tr>
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<tbody>
<tr>
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<td>16</td>
<td>1.4</td>
<td>2</td>
<td>2</td>
<td>4</td>
<td>8</td>
<td>4</td>
</tr>
<tr>
<td>4</td>
<td>32</td>
<td>2</td>
<td>4</td>
<td>8</td>
<td>16</td>
<td>64</td>
<td>16</td>
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<tr>
<td>8</td>
<td>48</td>
<td>2.8</td>
<td>8</td>
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<td>512</td>
<td>256</td>
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<td>16</td>
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<td>80</td>
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<td>1024</td>
<td>32,768</td>
<td>$4.2 \times 10^9$</td>
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<tr>
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<td>96</td>
<td>8</td>
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<td>384</td>
<td>4,096</td>
<td>262,144</td>
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<td>112</td>
<td>11</td>
<td>128</td>
<td>896</td>
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<td>$3.4 \times 10^{38}$</td>
</tr>
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<td>$\vdots$</td>
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<td>$\vdots$</td>
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<td>$\vdots$</td>
<td>$\vdots$</td>
</tr>
<tr>
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<td>160</td>
<td>32</td>
<td>1,024</td>
<td>10,240</td>
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<td>$\vdots$</td>
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</tr>
<tr>
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<td>$6.7 \times 10^{315,652}$</td>
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</tbody>
</table>
Some Intuition on Meaning of Growth

- How big a problem can you solve in a given time?
- In the following table, left column shows time in microseconds to solve a given problem as a function of problem size \( N \).
- Entries show the size of problem that can be solved in a second, hour, month (31 days), and century, for various relationships between time required and problem size.

- \( N = \) problem size

<table>
<thead>
<tr>
<th>Time (( \mu )sec) for problem size ( N )</th>
<th>( \log N )</th>
<th>( N )</th>
<th>( N \log N )</th>
<th>( N^2 )</th>
<th>( N^3 )</th>
<th>( 2^N )</th>
<th>1 second</th>
<th>1 hour</th>
<th>1 month</th>
<th>1 century</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \log N )</td>
<td>10(^{3000000} )</td>
<td>10(^{10000000000} )</td>
<td>10(^{8\cdot10^{11}} )</td>
<td>10(^{10^{14}} )</td>
<td>63000</td>
<td>1.3 \cdot 10^8</td>
<td>7.4 \cdot 10^{10}</td>
<td>6.9 \cdot 10^{13}</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( N )</td>
<td>10^6</td>
<td>3.6 \cdot 10^9</td>
<td>2.7 \cdot 10^{12}</td>
<td>3.2 \cdot 10^{15}</td>
<td>1000</td>
<td>60000</td>
<td>1.6 \cdot 10^6</td>
<td>5.6 \cdot 10^7</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( N \log N )</td>
<td>63000</td>
<td>1.3 \cdot 10^8</td>
<td>7.4 \cdot 10^{10}</td>
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<td>1500</td>
<td>14000</td>
<td>150000</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( N^2 )</td>
<td>1000</td>
<td>60000</td>
<td>1.6 \cdot 10^6</td>
<td>5.6 \cdot 10^7</td>
<td>100</td>
<td>1500</td>
<td>14000</td>
<td>150000</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( N^3 )</td>
<td>100</td>
<td>1500</td>
<td>14000</td>
<td>150000</td>
<td>20</td>
<td>32</td>
<td>41</td>
<td>51</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( 2^N )</td>
<td>20</td>
<td>32</td>
<td>41</td>
<td>51</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
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</tr>
</tbody>
</table>
Using the Notation

- Can use this order notation for any kind of real-valued function.
- We will use them to describe cost functions. Example:

  ```java
  /** Find position of X in list L, or -1 if not found. */
  int find(List L, Object X) {
      int c;
      for (c = 0; L != null; L = L.next, c += 1)
          if (X.equals(L.head)) return c;
      return -1;
  }
  ```

- Choose representative operation: number of `equals` tests.
- If \( N \) is length of \( L \), then loop does at most \( N \) tests: worst-case time is \( N \) tests.
- In fact, total # of instructions executed is roughly proportional to \( N \) in the worst case, so can also say worst-case time is \( O(N) \), regardless of units used to measure.
- Use \( N > M \) provision (in defn. of \( O(\cdot) \)) to ignore empty list.
Be Careful

• It’s also true that the worst-case time is $O(N^2)$, since $N \in O(N^2)$ also: Big-Oh bounds are loose.

• The worst-case time is $\Omega(N)$, since $N \in \Omega(N)$, but that does not mean that the loop always takes time $N$, or even $K \cdot N$ for some $K$.

• Instead, we are just saying something about the function that maps $N$ into the largest possible time required to process any array of length $N$.

• To say as much as possible about our worst-case time, we should try to give a $\Theta$ bound: in this case, we can: $\Theta(N)$.

• But again, that still tells us nothing about best-case time, which happens when we find $X$ at the beginning of the loop. Best-case time is $\Theta(1)$. 
Effect of Nested Loops

- Nested loops often lead to polynomial bounds:

```java
for (int i = 0; i < A.length; i += 1)
    for (int j = 0; j < A.length; j += 1)
        if (i != j && A[i] == A[j])
            return true;
return false;
```

- Clearly, time is $O(N^2)$, where $N = A.length$. **Worst-case time is $\Theta(N^2)$**.

- Loop is inefficient though:

```java
for (int i = 0; i < A.length; i += 1)
    for (int j = i+1; j < A.length; j += 1)
        if (A[i] == A[j]) return true;
return false;
```

- Now worst-case time is proportional to

$$N - 1 + N - 2 + \ldots + 1 = N(N - 1)/2 \in \Theta(N^2)$$

(so asymptotic time unchanged by the constant factor).
Recursion and Recurrences: Fast Growth

• Silly example of recursion. In the worst case, both recursive calls happen:

```java
/** True iff X is a substring of S */
boolean occurs(String S, String X) {
    if (S.equals(X)) return true;
    if (S.length() <= X.length()) return false;
    return
        occurs(S.substring(1), X) ||
        occurs(S.substring(0, S.length()-1), X);
}
```

• Define $C(N)$ to be the worst-case cost of $\text{occurs}(S,X)$ for $S$ of length $N$, $X$ of fixed size $N_0$, measured in # of calls to $\text{occurs}$. Then

$$C(N) = \begin{cases} 
1, & \text{if } N \leq N_0, \\
2C(N - 1) + 1 & \text{if } N > N_0 
\end{cases}$$

• So $C(N)$ grows exponentially:

$$C(N) = 2C(N - 1) + 1 = 2(2C(N - 2) + 1) + 1 = \ldots = 2\underbrace{(\ldots 2 \cdot 1 + 1)}_{N-N_0} + \ldots + 1 = 2^{N-N_0+1} - 1 \in \Theta(2^N)$$
Binary Search: Slow Growth

/** True X iff is an element of S[L .. U]. Assumes
 * S in ascending order, 0 <= L <= U-1 < S.length. */
boolean isIn(String X, String[] S, int L, int U) {
    if (L > U) return false;
    int M = (L+U)/2;
    int direct = X.compareTo(S[M]);
    if (direct < 0) return isIn(X, S, L, M-1);
    else if (direct > 0) return isIn(X, S, M+1, U);
    else return true;
}

• Here, worst-case time, $C(D)$, (as measured by # of calls to .compareTo),
depends on size $D = U - L + 1$.

• We eliminate $S[M]$ from consideration each time and look at half the
  rest. Assume $D = 2^k - 1$ for simplicity, so:

$$C(D) = \begin{cases} 
0, & \text{if } D \leq 0, \\
1 + C((D - 1)/2), & \text{if } D > 0.
\end{cases}$$

$$= 1 + 1 + \ldots + 1 + 0$$

$$= k = \lg(D + 1) \in \Theta(\lg D)$$
Another Typical Pattern: Merge Sort

List sort(List L) {
    if (L.length() < 2) return L;
    Split L into L0 and L1 of about equal size;
    L0 = sort(L0); L1 = sort(L1);
    return Merge of L0 and L1
}

- Assuming that size of L is $N = 2^k$, worst-case cost function, $C(N)$, counting just merge time (which is proportional to # items merged):

$$C(N) = \begin{cases} 
0, & \text{if } N < 2; \\
2C(N/2) + N, & \text{if } N \geq 2.
\end{cases}$$

\[
\begin{align*}
&= 2(2C(N/4) + N/2) + N \\
&= 4C(N/4) + N + N \\
&= 8C(N/8) + N + N + N \\
&= N \cdot 0 + \underbrace{N + N + \cdots + N}_{k=\lg N} \\
&= N \lg N
\end{align*}
\]

- In general, can say it’s $\Theta(N \lg N)$ for arbitrary $N$ (not just $2^k$).