Today:

- Pseudo-random Numbers (Chapter 11)
- What use are random sequences?
- What are “random sequences”?
- Pseudo-random sequences.
- How to get one.
- Relevant Java library classes and methods.
- Random permutations.
Why Random Sequences?

- Choose statistical samples
- Simulations
- Random algorithms
- Cryptography:
  - Choosing random keys
  - Generating streams of random bits (e.g., stream ciphers encrypt messages by xor'ing reproducible streams of pseudo-random bits with the bits of the message.)
- And, of course, games
What Is a “Random Sequence”?

• How about: “a sequence where all numbers occur with equal frequency”?
  - Like 1, 2, 3, 4, …?

• Well then, how about: “an unpredictable sequence where all numbers occur with equal frequency?”
  - Like 0, 0, 0, 1, 1, 2, 2, 2, 2, 3, 4, 4, 0, 1, 1, 1,…?

• Besides, what is wrong with 0, 0, 0, 0, … anyway? Can’t that occur by random selection?
Pseudo-Random Sequences

- Even if definable, a “truly” random sequence is difficult for a computer (or human) to produce.

- For most purposes, need only a sequence that satisfies certain statistical properties, even if deterministic.

- Sometimes (e.g., cryptography) need sequence that is hard or impractical to predict.

- **Pseudo-random sequence**: deterministic sequence that passes some given set of statistical tests.

- For example, look at lengths of runs: increasing or decreasing contiguous subsequences.

- Unfortunately, statistical criteria to be used are quite involved. For details, see Knuth.
Generating Pseudo-Random Sequences

- Not as easy as you might think.
- Seemingly complex jumbling methods can give rise to bad sequences.
- **Linear congruential method** is a simple method used by Java:
  \[
  X_0 = \text{arbitrary seed} \\
  X_i = (aX_{i-1} + c) \mod m, \quad i > 0
  \]

- Usually, \( m \) is large power of 2.
- For best results, want \( a \equiv 5 \mod 8 \), and \( a, c, m \) with no common factors.
- This gives generator with a **period of \( m \)** (length of sequence before repetition), and reasonable **potency** (measures certain dependencies among adjacent \( X_i \)).
- Also want bits of \( a \) to “have no obvious pattern” and pass certain other tests (see Knuth).
- **Java uses** \( a = 25214903917, c = 11, m = 2^{48} \), to compute 48-bit pseudo-random numbers. It’s good enough for many purposes, but not **cryptographically secure**.
What Can Go Wrong (I)?

- Short periods, many impossible values: E.g., $a$, $c$, $m$ even.
- Obvious patterns. E.g., just using lower 3 bits of $X_i$ in Java’s 48-bit generator, to get integers in range 0 to 7. By properties of modular arithmetic,

$$X_i \mod 8 = (25214903917X_{i-1} + 11 \mod 2^{48}) \mod 8$$
$$= (5(X_{i-1} \mod 8) + 3) \mod 8$$

so we have a period of 8 on this generator; sequences like

$$0, 1, 3, 7, 1, 2, 7, 1, 4, \ldots$$

are impossible. This is why Java doesn’t give you the raw 48 bits.
What Can Go Wrong (II)?

Bad potency leads to bad correlations.

- The infamous IBM generator RANDU: \( c = 0, \ a = 65539, \ m = 2^{31} \).
- When RANDU is used to make 3D points: \( (X_i/S, X_{i+1}/S, X_{i+2}/S) \), where \( S \) scales to a unit cube, . . .
- . . . points will be arranged in parallel planes with voids between. So “random points” won’t ever get near many points in the cube:

![Diagram showing parallel planes with voids between points.](https://commons.wikimedia.org/wiki/thumb/3832343)


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Additive Generators

• Additive generator:

\[ X_n = \begin{cases} 
\text{arbitrary value,} & n < 55 \\
(X_{n-24} + X_{n-55}) \mod 2^e, & n \geq 55 
\end{cases} \]

• Other choices than 24 and 55 possible.
• This one has period of \( 2^f(2^{55} - 1) \), for some \( f < e \).
• Simple implementation with circular buffer:

```c
i = (i+1) % 55;
X[i] += X[(i+31) % 55];  // Why +31 (55-24) instead of -24?
return X[i];  /* modulo \( 2^{32} \) */
```

• where \( X[0 \ldots 54] \) is initialized to some “random” initial seed values.
Cryptographic Pseudo-Random Number Generators

- The simple form of linear congruential generators means that one can predict future values after seeing relatively few outputs.

- Not good if you want *unpredictable* output (think on-line games involving money or randomly generated keys for encrypting your web traffic.)

- A cryptographic pseudo-random number generator (CPRNG) has the properties that
  - Given $k$ bits of a sequence, no polynomial-time algorithm can guess the next bit with better than 50% accuracy.
  - Given the current state of the generator, it is also infeasible to reconstruct the bits it generated in getting to that state.
Cryptographic Pseudo-Random Number Generator Example

- Start with a good *block cipher*—an encryption algorithm that encrypts blocks of \( N \) bits (not just one byte at a time as for Enigma). AES is an example.
- As a seed, provide a key, \( K \), and an initialization value \( I \).
- The \( j^{th} \) pseudo-random number is now \( E(K, I + j) \), where \( E(x, y) \) is the encryption of message \( y \) using key \( x \).
Adjusting Range and Distribution

- Given raw sequence of numbers, \( X_i \), from above methods in range (e.g.) 0 to \( 2^{48} \), how to get uniform random integers in range 0 to \( n - 1 \)?

- If \( n = 2^k \), is easy: use top \( k \) bits of next \( X_i \) (bottom \( k \) bits not as “random”)

- For other \( n \), be careful of slight biases at the ends. For example, if we compute \( X_i / (2^{48}/n) \) using all integer division, and if \( (2^{48}/n) \) gets rounded down, then you can get \( n \) as a result (which you don’t want).

- If you try to fix that by computing \( (2^{48}/(n - 1)) \) instead, the probability of getting \( n - 1 \) will be wrong.
Adjusting Range (II)

• To fix the bias problems when \( n \) does not evenly divide \( 2^{48} \), Java throws out values after the largest multiple of \( n \) that is less than \( 2^{48} \):

```java
/** Random integer in the range 0 .. n-1, n>0. */
int nextInt(int n) {
    long X = next random long (0 ≤ X < 2^{48});
    if (n is \( 2^k \) for some \( k \))
        return top \( k \) bits of \( X \);

    int MAX = largest multiple of \( n \) that is < \( 2^{48} \);
    while (X ≥ MAX)
        X = next random long (0 ≤ X < 2^{48});
    return X \( i \) / (MAX/n);
}
```
Arbitrary Bounds

• How to get arbitrary range of integers \((L \text{ to } U)\)?

• To get random float, \(x\) in range \(0 \leq x < d\), compute

  \[
  \text{return } d \times \text{nextInt}(1\ll24) / (1\ll24);
  \]

• Random double a bit more complicated: need two integers to get enough bits.

  \[
  \text{long bigRand = ((long) nextInt}(1\ll26) \ll 27) + (\text{long) nextInt}(1\ll27);
  \text{return } d \times \text{bigRand} / (1L \ll 53);
  \]
Generalizing: Other Distributions

• Suppose we have some desired probability distribution function, and want to get random numbers that are distributed according to that distribution. How can we do this?

• Example: the normal distribution:

\[ P(Y \leq X) \]

-2 -1 0 1 2

\( X \)

- Curve is the desired probability distribution. \( P(Y \leq X) \) is the probability that random variable \( Y \) is \( \leq X \).
Other Distributions

**Solution:** Choose $y$ uniformly between 0 and 1, and the corresponding $x$ will be distributed according to $P$.

$$P(X \leq Y)$$
Java Classes

- Math.random(): random double in \([0..1]\).

- Class java.util.Random: a random number generator with constructors:
  
  Random() generator with “random” seed (based on time).
  Random(seed) generator with given starting value (reproducible).

- Methods

  next\((k)\) \(k\)-bit random integer
  nextInt\((n)\) int in range \([0..n]\).
  nextLong() random 64-bit integer.
  nextBoolean(), nextFloat(), nextDouble() Next random values of other primitive types.
  nextGaussian() normal distribution with mean 0 and standard deviation 1 (“bell curve”).

- Collections.shuffle\((L, R)\) for list \(L\) and Random \(R\) permutes \(L\) randomly (using \(R\)).
Shuffling

- A **shuffle** is a random permutation of some sequence.
- Obvious dumb technique for sorting $N$-element list:
  - Generate $N$ random numbers
  - Attach each to one of the list elements
  - Sort the list using random numbers as keys.
- Can do quite a bit better:

  ```java
  void shuffle(List L, Random R) {
      for (int i = L.size(); i > 0; i -= 1)
          swap element i-1 of L with element R.nextInt(i) of L;
  }
  ```

- Example:

<table>
<thead>
<tr>
<th>Swap items</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Start</td>
<td>♠A</td>
<td>♠2</td>
<td>♠3</td>
<td>♠A</td>
<td>♠2</td>
<td>♠3</td>
</tr>
<tr>
<td>5 $\iff$ 1</td>
<td>♠A</td>
<td>3♥</td>
<td>3♠</td>
<td>♠A</td>
<td>2♥</td>
<td>2♠</td>
</tr>
<tr>
<td>4 $\iff$ 2</td>
<td>♠A</td>
<td>3♥</td>
<td>2♥</td>
<td>♠A</td>
<td>3♠</td>
<td>2♠</td>
</tr>
</tbody>
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<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>3 $\iff$ 3</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2 $\iff$ 0</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1 $\iff$ 0</td>
<td></td>
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</tr>
</tbody>
</table>
Random Selection

- Same technique would allow us to select $N$ items from list:

  /** Permute L and return sublist of K>=0 randomly
   * chosen elements of L, using R as random source. */
  List select(List L, int k, Random R) {
    for (int i = L.size(); i+k > L.size(); i -= 1)
      swap element i-1 of L with element
        R.nextInt(i) of L;
    return L.sublist(L.size()-k, L.size());
  }

- Not terribly efficient for selecting random sequence of $K$ distinct integers from $[0..N)$, with $K \ll N$. 
Alternative Selection Algorithm (Floyd)

/** Random sequence of K distinct integers * from 0..N-1, 0<=K<=N. */
IntList selectInts(int N, int K, Random R)
{
    IntList S = new IntList();

    for (int i = N-K; i < N; i += 1) {
        // All values in S are < i
        int s = R.randInt(i+1); // 0 <= s <= i < N
        if (s == S.get(j) for some j)
            // Insert value i (which can’t be there
            // yet) after the s (i.e., at a random
            // place other than the front)
            S.add(j+1, i);
        else
            // Insert random value s at front
            S.add(0, s);
    }

    return S;
}

Example

<table>
<thead>
<tr>
<th>i</th>
<th>s</th>
<th>S</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>4</td>
<td>[4]</td>
</tr>
<tr>
<td>6</td>
<td>2</td>
<td>[2, 4]</td>
</tr>
<tr>
<td>7</td>
<td>5</td>
<td>[5, 2, 4]</td>
</tr>
<tr>
<td>8</td>
<td>5</td>
<td>[5, 8, 2, 4]</td>
</tr>
<tr>
<td>9</td>
<td>4</td>
<td>[5, 8, 2, 4, 9]</td>
</tr>
</tbody>
</table>

selectRandomIntegers(10, 5, R)