INFINITY, SELF-REFERENCE, AND UNDECIDABILITY

GIVEN 2 SETS A, B. HOW DO WE KNOW IF THEY HAVE THE SAME NUMBER OF ELEMENTS.

f: A → B: 1→a, 2→b, 3→c

$|A| = |B|$ iff THERE IS A BIJECTION $f: A \to B$

f IS 1-1 : $x \neq y \Rightarrow f(x) \neq f(y)$

f IS ONTO IF $\forall y \; \exists x : f(x) = y$

} BIJECTION IS BOTH 1-1 AND ONTO

$N = \{0, 1, 2, \cdots\}$ COUNTABLE

$Z^- = \{0, -1, -2, \cdots\}$ $0 \to 0$ $1 \to -1 \cdots$

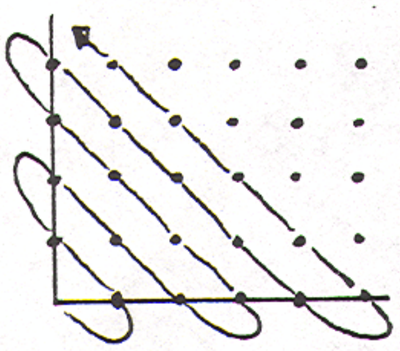
$E = \{0, 2, 4, \cdots\}$ $f: N \to E$ $f(x) = 2x$

$Z = \{\cdots -2, -1, 0, 1, 2, \cdots\} = \{0, 1, -1, 2, -2, \cdots\}$

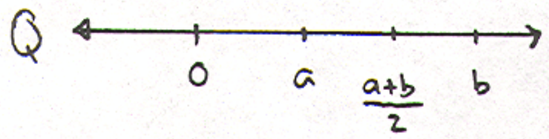
$f: N \to Z$ $f(x) = \begin{cases} \frac{x}{2} & \text{IF } x \text{ IS EVEN} \\ -\frac{(x-1)}{2} - 1 & \text{IF } x \text{ IS ODD} \end{cases}$

$N = \{0, 1, 2, 3, 4, \cdots\}$

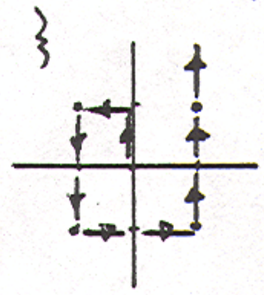
$Z = \{0, -1, 1, -2, 2, \cdots\}$



$Q = \{(x, y) : x, y \in N\}$



$Q = \left\{\frac{x}{y} : x, y \in Z\right\}$



REAL NUMBERS R 2.31⋯ π e ⟷

THERE ARE MORE REAL NUMBERS THAN NATURAL NUMBERS.

CANTORS DIAGONALIZATION ARGUMENT

SUPPOSE FOR CONTRADICTION THAT THE REAL'S ARE COUNTABLE SAY $R : [0, 1]$

0 ⟷ 0.52149 5→0

1 ⟷ 0.21327

2 ⟷ 0.41621 75→5

3 ⟷ 0.29473

⋮

0.0555⋯ IS THIS NUMBER ON THE LIST?

IF $r = j^{th}$ NUMBER ON THE LIST, THEN THE j^{th} DIGIT WOULD BE DIFFERENT. ∴ IT IS NOT ON THE LIST BUT SHOULD BE!

$f(x) = x$ FOR $f: N \to R$ f IS 1-1

HIGHER ORDERS OF INFINITY

S, ANY SET $S = \{1, 2, 3, \cdots\}$

$P(S)$ POWER SET OF S = SET OF ALL SUBSETS OF S $P(S) = \{\phi, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}$

$|S| = k$ $|P(S)| = ? = \binom{k}{0} + \binom{k}{1} + \cdots + \binom{k}{k} = 2^k$

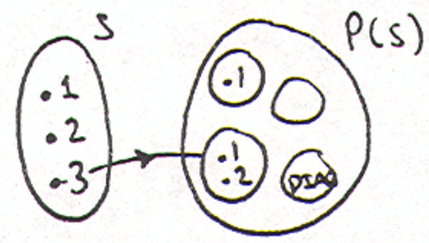
EITHER CHOOSE OR NOT 1ST ELEMENT — 2 CHOICES

" " " " 2ND " — 2 CHOICES

⋯

" " " " k ELEMENTS

IF S IS INFINITE IS $|S| = |P(S)|$



SUPPOSE THERE IS BIJECTION FROM S TO $P(S)$ $f: S \to P(S)$

$DIAG = \underbrace{\{x \in S : x \notin f(x)\}}_{\subseteq S}$ IS $DIAG = f(y)$ FOR ANY $y \in S$?

$y \in DIAG$? IF YES - CANNOT BE!

IF NO - THEN IT SHOULD!

} CONTRADICTION!

SUBSETS OF S

1110⋯

$\aleph_0$ ALEPH NULL (COUNTABLE)

$\aleph_1$ ALEPH ONE

⋮ ⋮