Problem Set 3

1. **Simple Induction** Prove that for every $n \geq 2$, $\sqrt{1+\sqrt{1+\cdots\sqrt{1}}}$ (n square roots) is irrational.

2. **Strengthen induction hypothesis** Prove that for all $n \geq 1$, all the entries of the matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^n$ are bounded above by n.

3. **Proofs to Grade**
Grade the following induction proofs. If you think the proof is incorrect, please explain exactly what is wrong with the structure or the reasoning in the *proof.* Remember, simply saying that the claim (or the induction hypothesis is false) is *not* a justification.

   (a) **Claim:** $(\forall n \in \mathcal{N})(n^2 \leq n)$. **Proof:**

   i) Base Case: When $n = 1$, the statement is $1^2 \leq 1$ whcih is true.

   ii) Induction hypothesis: Assume that $k^2 \leq k$.

   ii) Inductive step: We need to show that

   $$(k+1)^2 \leq k+1$$

   Working backwards we see that:

   $$(k+1)^2 \leq k+1k^2+2k+1 \leq k+1k^2+2k \leq kk^2 \leq k$$

   So we get back to our original hypothesis which is assumed to be true. Hence, for every $n \in \mathcal{N}$ we know that $n^2 \leq n$. ♠

   (b) **Claim:** $(\forall n \in \mathcal{N})(7^n = 1)$.
   Proof: (uses strong induction)
   Base Case: Certainly $7^0 = 1$.

   Induction hypothesis: Assume that $7^j = 1$ for all $0 \leq j \leq k$.

   Inductive step: We need to prove that $7^{k+1} = 1$. But,

   $$7^{k+1} = \frac{(7^k \cdot 7^k)}{7^{k-1}} = \frac{(1 \cdot 1)}{1} = 1$$

   Hence, by the Principle of Strong Induction, $7^m = 1$ for all $m \in \mathcal{N}$. ♡

   (c) **Claim:** For all natural numbers $n \geq 4$, $2^n < n!$.
   Proof: $2^4 = 16$ and $4! = 24$, so the statement is true for $n = 4$.
   Assume that $2^n < n!$ for some $n \in \mathcal{N}$. Then $2^{n+1} = 2(2^n) < 2(n!) \leq (n+1)(n!) = (n+1)!$, so $2^{n+1} < (n+1)!$. By the principle of mathematical induction, the statement is true for all $n \geq 4$. ♣

(d) **Claim:** All natural numbers are equal.

Proof: It is sufficient to show that for any two natural numbers A and B, A = B. Further, it is sufficient to show that for all $N \geq 0$, if A and B satisfy MAX(A, B) = N then A = B.

Proceed by induction on N.

Base case: If N = 0, then A and B, being natural numbers, must both be 0. So A = B.

Induction Step: Assume that the theorem is true for some value k. Take A and B with MAX(A, B) = k+1. Then MAX((A-1), (B-1)) = k. And hence (A-1) = (B-1). Consequently, A = B. $\diamondsuit$

4. Consider the following recursively program for a function f:

```
function f(n)
    if n = 0 return 0
    else return(f(n − 1) + 2n)
end function
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Prove by induction that $f(n) = n^2 + n$.

5. Examples for stable marriage algorithm.

(a) For the following instance of the stable marriage problem:

Men's preference list:

1	B	A	C
2	C	B	A
3	A	C	B

Women's preference list:

A	1	2	3
B	2	3	1
C	3	1	2

i. List the rogue couples in the pairing $(1, C), (2, B), (3, A)$.

ii. List all the stable pairings.

(b) Run the traditional propose and reject algorithm on the following example.

Men's preference list:

1	A	B	C	D
2	B	C	A	D
3	C	A	B	D
4	A	B	C	D

Women's preference list:

A	2	3	4	1
B	3	4	1	2
C	4	1	2	3
D	1	2	3	4